Optimal Thresholding Quantizer Maximizing Mutual Information of Discrete Multiple-Input Continuous One-Bit Output Quantization

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***Abstract*—In this paper, we consider the problem of one-bit (two-level) output-quantization maximizing mutual information between quantizer-output and channel-input using a single threshold for a discrete signal that is corrupted by a continuous additive noise. A necessary condition is constructed for which the thresholding quantizer is optimal. In addition, we show that if the distribution of the additive noise satisfies a mild condition, then a global optimal threshold can be found efficiently via a modified fixed-point algorithm.**

Keyword: quantization, mutual information, threshold.

I. Introduction

Quantization aims to map a real value signal to a finite quantized-set that maximizes or minimizes an objective function. The mean square error (MSE) between the original signal and the quantized signal is the most common objective function that was used in various communication systems [1], [2]. Recently, motivated by designing of low-density parity-check (LDPC) codes and polar codes [3]–[5], finding the optimal quantizer that maximizes the mutual information between the input signal and the quantized-output is of great interest. Consequently, in recent years, there is a rich literature on finding such quantizers [6]–[18]. It is worth noting that the problem of designing quantizer maximizing mutual information is closely related to the problem of noisy source quantization [19]–[21]. In addition, finding the optimal quantizer that maximizes mutual information is an extremely hard problem even with the one-bit (binary) quantization. In a special case where the input is binary, Kurkoski and Yagi showed that the optimal quantizer separates the posterior distribution into the contiguous interval [7], [22]. Thus, a global optimal quantizer can be found efficiently using dynamic programming technique and its variants [7], [12]. For the larger size of the input alphabet, the beautiful result in [7], [22] is inapplicable, and finding a global optimal quantizer requires a naive exhaustive search with exponential time complexity [23]. Of course, an exhaustive search approach will quickly become computationally intractable even for the modest values of the number of input and the number of data. To overcome this inconvenience, many works consider a sub-class quantizer called thresholding quantizer which allows determining a global optimal quantizer efficiency for non-binary input channels [9], [15]–[17], [22], [24]–[29]. An important property of the thresholding quantizer is that it maps every disjoint interval of input to a distinct quantized-output using scalar thresholds as the boundaries of these intervals. The advantage of this thresholding quantizer is that it has a simple circuit implementation which is similar to the classic scalar quantizer. This special structure makes thresholding quantizer is suitable to decode the Pulse Amplitude Modulation signals (PAM) where the output symbols are resolved based on the magnitude of the received signal. However, it is worth noting that thresholding quantizers might not be truly optimal quantizers for maximizing mutual information, i.e., the real optimal quantizer that maximizes the mutual information between the input and the quantized-output might map the output to the disjoint intervals [22], [29].

Based on the structure of the thresholding quantizer, many algorithms were proposed to find the optimal quantization. In [25], the author proposed a heuristic near-optimal quantization algorithm that alternatively maximizes the mutual information for a given quantizer and minimizes the probability of error for a given input distribution. However, this algorithm only works well when the signal-to-noise ratio of the channel is high. For 1-bit quantization of general additive channels, Mathar and Dorpinghaus gave a condition for which the threshold is optimal [16]. They also pointed out that the optimal threshold must be between two support points and the optimal threshold might not be unique. However, the work of Mathar and Dorpinghaus [16] still requires an exhaustive search between two support points for determining a global optimal threshold. In [22], Kurkoski and Yagi considered a single-bit quantization of binary input continuous-output channel and showed that the optimal quantizer corresponds to a scalar threshold in the space of the posterior distribution. Using the result of Kurkoski and Yagi, one can find the optimal quantizer by exhaustive search a scalar threshold over the posterior distribution. The work of Kurkoski and Yagi is further investigated in [10] and a global optimal quantizer can be found via a modified fixed-point algorithm. For multiple-input and multiple-output thresholding quantizer, if the channel satisfies pre-determined conditions, the global optimal quantizer can be found efficient via the famous dynamic programming and SMAWK algorithm [15], [27], [28].

Although existing efficient algorithms to determine the optimal thresholding quantizer in some special cases, the most important case of discrete multiple-input continuous one-bit single threshold quantization still requires the exhaustive search [16] which fundamentally limits the performance of designing the optimal quantizer for one-bit quantization. In this paper, we investigate the problem of finding the optimal quantizer for discrete multiple-input continuous one-bit output single-threshold quantization that maximizes the mutual information between the input and the quantized-output. A necessary condition for optimal thresholds is constructed. Based on this optimality condition, if the additive noise distribution satisfies a mild condition, an efficient algorithm is proposed to determine the optimal quantizers. To that end, the most related works of this paper are that of Mathar and Dorpinghaus [16] and Nguyen [10]. Particularly, our setting in this paper is similar to one in [16], however, we assume that the input distribution is given which allows a global optimal quantizer can be found efficient via a modified fixed-point algorithm. On the other hand, the modified fixed-point algorithm has been successfully used in [10] for finding a global optimal quantizer of discrete binary-input continuous binary-output channels. Although the algorithmic approach is similar, the established results in this paper are not straightforwardly constructed from [10]. Indeed, due to the larger size of the input alphabet, all the proofs and results require a deeper investigation which contributes to the novelty of this paper.

II. Problem Formulation

Consider a channel having the input alphabet set X consisting of N transmitted symbols $X = \{x_1, x_2, \ldots, x_N\}$ with a given input p.m.f $\mathbf{p}_X = \{p_1, p_2 \ldots, p_N\}$, $x_i \in \mathbb{R}$ and $x_{i-1} < x_i$, $\forall i$. X is transmitted over a noisy channel having a continuous additive noise W to produce a continuous received-output $Y = X + W$. Due to the continuous additive noise, the received signal $y \in Y = \mathbb{R}$ is modeled via the conditional density function $p_{Y|X}(y|x_i) = \phi_i(y)$, $i = 1, 2, \ldots, N$. From the additive property, $\phi_i(y) = p_{Y|X}(y|x_i)$ and $\phi_j(y) = p_{Y|X}(y|x_j)$, $i \neq j$, are the shifting versions of each other. A quantizer $Q : Y \rightarrow Z$ is used to quantize the continuous output $y \in Y$ to a binary discrete quantized-output $Z = \{z_0, z_1\}$. Since the thresholding quantizer is used, quantizer Q is equivalent to a single threshold h such that:

$$Q(y) = \begin{cases} z_0 & \text{if } y < h, \\ z_1 & \text{otherwise.} \end{cases}$$

One wants to design an optimal quantizer Q^* which in turn corresponds to an optimal threshold h^* that maximizes the mutual information between the input and the quantized-output $I(X; Z)$:

$$h^* = \underset{h}{\operatorname{argmax}}\, I(X; Z). \tag{1}$$

III. Optimality Condition and Algorithm

In this section, an optimality condition for the optimal threshold is constructed. Based on this optimality condition, a fast algorithm is proposed to find a global optimal quantizer. All the results in this paper assume that the conditional density function $\phi_i(y)$, $i = 1, 2, \ldots, N$, is continuous, positive, and differentiable everywhere. Many well-known distributions such as normal distribution, exponential distribution, satisfy these requirements. We first begin with some notations and definitions.

A. Notations and definitions

1) For a given threshold h, let $p(z_0|x_i)$ and $p(z_1|x_i)$ denote the conditional distribution of the quantized-output z_0 and z_1 given the input x_i, respectively. Specifically,

$$p(z_0|x_i) = \int_{-\infty}^{h} \phi_i(y) dy, \tag{2}$$

$$p(z_1|x_i) = \int_{h}^{+\infty} \phi_i(y) dy = 1 - p(z_0|x_i). \tag{3}$$

2) For a given threshold h, the channel matrix is defined by:

$$A = \begin{pmatrix} p(z_0|x_1) & p(z_1|x_1) \\ \vdots & \vdots \\ p(z_0|x_N) & p(z_1|x_N) \end{pmatrix}$$

3) Let q_0 and q_1 denote the probability of the quantized-output z_0 and z_1, respectively. Specifically,

$$q_0 = \sum_{i=1}^{N} p_i p(z_0|x_i), \tag{4}$$

$$q_1 = \sum_{i=1}^{N} p_i p(z_1|x_i) = 1 - q_0. \tag{5}$$

4) Let $p(x_i|z_0)$ and $p(x_i|z_1)$ denote the conditional distribution of x_i given z_0 and z_1, respectively. Specifically,

$$p(x_i|z_0) = \frac{p_i p(z_0|x_i)}{\sum_{i=1}^{N} p_i p(z_0|x_i)}, \tag{6}$$

$$p(x_i|z_1) = \frac{p_i p(z_1|x_i)}{\sum_{i=1}^{N} p_i p(z_1|x_i)}. \tag{7}$$

5) Let $\mathbf{v}(x|y)$ denote the conditional distribution vector of the input $x_1, x_2, \ldots, x_N$ given y. Specifically,

$$\mathbf{v}(x|y) = [p(x_1|y), p(x_2|y), \ldots, p(x_N|y)], \tag{8}$$

$$p(x_i|y) = \frac{p_i \phi_i(y)}{\sum_{i=1}^{N} p_i \phi_i(y)}. \tag{9}$$

6) Let $\mathbf{v}(x|z_0)$ and $\mathbf{v}(x|z_1)$ denote the conditional distribution vector of the input $x_1, x_2, \ldots, x_N$ given the quantized-output z_0 and z_1, respectively. Specifically,

$$\mathbf{v}(x|z_0) = [p(x_1|z_0), p(x_2|z_0), \ldots, p(x_N|z_0)], \tag{10}$$

$$\mathbf{v}(x|z_1) = [p(x_1|z_1), p(x_2|z_1), \ldots, p(x_N|z_1)]. \tag{11}$$

Definition 1. (KL-divergence.) The Kullback-Leibler (KL) divergence of two probability vectors $\mathbf{a} =$

$[a_1, a_2, \ldots, a_N]$ and $\mathbf{b} = [b_1, b_2, \ldots, b_N]$ of the same outcome set N is defined by:

$$D(\mathbf{a}||\mathbf{b}) = \sum_{i=1}^{N} a_i \log(\frac{a_i}{b_i}). \quad (12)$$

Definition 2. (Dominated conditional distribution channel.) A channel is dominated conditional distribution channel if all the distributions $\phi_i(y)$ satisfy:

$$\frac{\phi_i(y)}{\phi_j(y)} > \frac{\phi_i(y')}{\phi_j(y')}, \quad (13)$$

for $\forall\ i \leq j$ and $y \leq y'$.

In practice, the inequality (13) is not too restricted. Indeed, the inequality (13) holds for a variety of common noise distributions such as normal distribution, exponential distribution, gamma distribution, and more generally, all log-concave distributions (please see Corollary 1 and 2 [10]).

B. Optimality conditions

Theorem 1. (**Optimality condition.**) Each optimal quantizer Q^* (local or global) corresponds to an optimal threshold h^* such that:

$$D(\mathbf{v}(x|h^*)||\mathbf{v}(x|z_0)) = D(\mathbf{v}(x|h^*)||\mathbf{v}(x|z_1)). \quad (14)$$

Proof: Please see the proof in Theorem 1 [28]. ∎

Theorem 1 confirms the result in [6] which stated that designing optimal quantizers maximizing mutual information is equivalent to finding optimal clusters minimizing KL-divergence distortion. Since the optimal threshold h^* acts as an optimal boundary between z_0 and z_1, the KL-divergences from h^* to the "centroids" of z_0 and z_1 must be equal. Unluckily, Theorem 1 is not a sufficient condition and the iterative algorithm in [6] is only possible to find the locally optimal thresholds. To guarantee a globally optimal solution, one still needs to perform an exhaustive search over $h \in Y = \mathbb{R}$.

Theorem 2. (**Existence of the optimal threshold**.) There exists an optimal threshold h^* that maximizes the mutual information $I(X;Z)$. Moreover, h^* is finite.

Proof: Please see the proof in Appendix A. ∎

Definition 3. For given $\mathbf{v}(x|z_0)$ and $\mathbf{v}(x|z_1)$ which correspond to a threshold h, let $f(h)$ be a mapping $f(h) : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$D(\mathbf{v}(x|f(h))||\mathbf{v}(x|z_0)) = D(\mathbf{v}(x|f(h))||\mathbf{v}(x|z_1)). \quad (15)$$

Noting that for a given h and therefore, given z_0 and z_1, $f(h)$ acts as an optimal separating threshold between z_0 and z_1.

Theorem 3. Consider a dominated conditional distribution channel, for given $\mathbf{v}(x|z_0)$ and $\mathbf{v}(x|z_1)$ which correspond to a threshold h, then:

1) There exists a finite and unique $f(h)$ satisfies (15).
2) $f(h)$ is a non-decreasing function of h.

Proof: Please see the proof in Appendix B. ∎

The results in Theorem 3 play a key role in the construction of our modified fixed-point algorithm. Next, Theorem 4, Lemma 1, and Theorem 5 can be constructed based on Theorem 3. Noting that the proofs of Theorem 4, Lemma 1, and Theorem 5 in this paper are similar to the proofs of Theorem 3, Lemma 5, and Theorem 5 in [10], respectively. However, for the convenience of the reader, we still sketch the proofs of these theorems.

Theorem 4. For any dominated conditional distribution channel, if h^* is an optimal threshold, then $f(h^*) = h^*$.

Proof: From (14), (15), both h^* and $f(h^*)$ satisfy (15). From Theorem 3, (15) has a unique solution. Thus, $h^* = f(h^*)$. ∎

Theorem 4 states that the optimal threshold h^* must be a fixed point of $f(h)$. Therefore, h^* can be found via an iterative fixed-point algorithm with an initial threshold h^0. The iterative fixed-point algorithm then will update h^{i+1} from h^i using $h^{i+1} = f(h^i)$. Next, we show some interesting properties of $f(h)$.

Lemma 1. If $h^{i+1} = f(h^i)$, then the sequence h^i must converge to a fixed point h^* from any initial point h^0.

Proof: From Theorem 3, $f(h)$ is a non-decreasing function. Thus, the sequence produced by $h^{i+1} = f(h^i)$, starting from any h^0 must be monotone, i.e., $h^{i+1} \geq h^i$ $\forall i$, or $h^{i+1} \leq h^i$ $\forall i$. Particularly, if $h^1 \leq h^0$, then $h^2 = f(h^1) \leq f(h^0) = h^1$, thus, $h^2 \leq h^1$. By induction method, if $h^1 \leq h^0$ then $h^{i+1} \leq h^i$, $\forall i$. Symmetrically, if $h^1 \geq h^0$, then $h^{i+1} \geq h^i$, $\forall i$. Thus, the sequence h^i is monotone. From Theorem 3, $f(h^i)$ generates a finite sequence or h^i is bounded. Thus, sequence h^i must convergence to a limit h^* such that $h^* = f(h^*)$. ∎

Theorem 5. For any initial point h^0, if $h^{i+1} = f(h^i)$ and $\lim_{i \to +\infty} h^i = h^*$, then there is no other solution h' such that $h' = f(h')$ between h_0 and h^*.

Proof: Let consider the case where $h^0 \leq h^*$ and suppose that there is a h' such that $h' = f(h')$ and $h^0 < h' < h^*$. Due to the sequence h^i is monotone, there exists an i such that $h^i < h' < h^{i+1}$. Now, since $f(h)$ is non-decreasing (Theorem 3), we have $h^{i+1} = f(h^i) \leq f(h') = h'$ which contradicts the assumption that $h' < h^{i+1}$. Thus, there is no other solution h' between h_0 and h^* if $h^0 \leq h^*$. Similarly, one can verify that there is no other solution h' in the interval (h^*, h^0) if $h^0 > h^*$. ∎

C. Outline of algorithm

From Theorem 2 and Theorem 3, it is possible to limit the searching range of h^* in a finite range $[-A, A]$, where A is a positive finite number. Next, based on the Theorem 4 and Lemma 1, an iterative fixed-point algorithm [30] can be used for finding the optimal threshold h^* by starting with a random number $h^0 \in [-A, A]$. However, this iterative algorithm only leads to a locally optimal solution. That said, if the equation $h = f(h)$ has more than one solution, we need a procedure capable of finding all the solutions of $h = f(h)$. Fortunately, Theorem 5 can help to find all the solutions of $h = f(h)$. A global optimal solution then can be chosen among these solutions.

Our procedure initiates using two starting points $h_l^0 = -A$ and $h_r^0 = A$. Suppose that the first starting point converges to h_l^*, and the second point converges to h_r^*. If $h_l^* = h_r^*$, then the procedure stops with $h^* = h_r^* = h_l^*$ being the optimal point due to Theorem 5 states that there is no solution of $h = f(h)$ in either $[-A, h^*]$ or $[h^*, A]$. Otherwise, if $h_l^* < h_r^*$, we need to search if existing other solutions in the interval (h_l^*, h_r^*). Therefore, the procedure initiates using another starting point $h^0 = (h_l^* + h_r^*)/2$. After this starting point converges to h_c^*, one needs to run the iterations over two intervals $(h_l^*, \min(h^0, h_c^*))$ and $(\max(h^0, h_c^*), h_r^*)$. If any of these intervals is nonempty, then the procedure must recursively repeat the previous steps until the whole interval $[-A, A]$ has been completely scanned. When all h^*'s are found, we pick the one that maximizes the mutual information.

It is worth noting that our procedure is based on the algorithm proposed in [10], [31]. Obviously that the modified fixed-point algorithm is much faster than an exhaustive search through all the values of $h \in [-A, A]$, however, it is still questionable to extend the modified fixed-point algorithm to more than one threshold quantizers even for scalar quantizers [31]. On the other hand, we believe that it is possible to recursive approximation the optimal multiple-threshold quantizers by a tree-structured of the optimal single-threshold quantizers [31].

IV. Conclusion

In this paper, the problem of finding the optimal single threshold for multiple-input one-bit output quantization maximizing mutual information is investigated. A necessary optimality condition is constructed for which the thresholding quantizer is optimal. In addition, we show that if the additive noise distribution satisfies a mild predetermined condition, then the optimal threshold can be found efficiently via a modified fixed-point algorithm.

Appendix

Due to the limited space, we either omit or only sketch the proofs. Please see the detailed proofs in our extension version.

A. *Proof of Theorem 2*

Existence of the optimal threshold: Each threshold h induces a channel matrix A which in turn produces the mutual information $I(X;Z)$. Since $\phi_i(y)$ is continuous, strictly positive and differentiable, mutual information $I(X;Z)$ is a continuous function of h i.e., $I(X;Z)_h$. Now, by using $h \to -\infty$ and $h \to +\infty$, it is obviously to show that one column of A must to reach to zero. Thus, if $h \to -\infty$ or $h \to +\infty$, $I(X;Z)_h \to 0$. By Rolle's theorem, there exist an optimal threshold h^* such that $\frac{\partial I(X;Z)}{\partial h^*} = 0$.

The finite of h^*: Let pick a finite h' that generates a channel matrix A having the mutual information $I(X;Z)_{h'} = \epsilon$ where $\epsilon > 0$. Obviously that any threshold h that induces $I(X;Z)_h < \epsilon$ is not a global optimal threshold. By contradiction method, assuming that existing an optimal threshold h^* which is infinite. Similar to the derivation in the previous proof, $I(X;Z)_{h^*} \to 0$ if $h \to \infty$, thus $I(X;Z)_{h^*} < \epsilon$. Therefore, $h \to \infty$ is not the optimal threshold. By contradiction, the optimal threshold is finite.

B. *Proof of Theorem 3*

Firstly, we begin with some definitions and supplemental results that will be used in proving Theorem 3. Noting that to prove all these results, we assume that the channel is dominated conditional distribution as stated in Definition 2.

Definition 4. We call a vector $\mathbf{a} = [a_1, \dots, a_N]$ majorizes a vector $\mathbf{b} = [b_1, \dots, b_N]$, denoted as $\mathbf{a} \succ \mathbf{b}$, if and only if $\sum_{i=1}^N a_i = \sum_{i=1}^N b_i$, and $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$, $\forall 1 \le n < N$.

For more information about majorization, please see [32].

Definition 5. Let $\mathbf{p}$ and $\mathbf{q}$ to be two N-dimensional vectors and:

$$\frac{\mathbf{p}}{\mathbf{q}} := \big[\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_N}{q_N}\big].$$

$\frac{\mathbf{p}}{\mathbf{q}}$ is an increasing N-vector if and only if $\frac{p_i}{q_i} < \frac{p_{i+1}}{q_{i+1}}$, $\forall i$.

The relationship between majorization, increasing vector, and KL-divergence can be viewed in [32], Corollary 3 and 4.

Proposition 1. Let $\mathbf{v_1}(x|z_0)$, $\mathbf{v_1}(x|z_1)$ and $\mathbf{v_2}(x|z_0)$, $\mathbf{v_2}(x|z_1)$ are the conditional distribution vectors of the input given the quantized-output $\{z_0, z_1\}$ corresponding to thresholds h_1 and h_2, respectively. If the channel is dominated conditional distribution and $h_1 < h_2$, then:

1) $\mathbf{v}(x|h_1) \succ \mathbf{v}(x|h_2)$.
2) $\mathbf{v_1}(x|z_0) \succ \mathbf{v}(x|h_1) \succ \mathbf{v_1}(x|z_1)$.
3) $\mathbf{v_2}(x|z_0) \succ \mathbf{v}(x|h_2) \succ \mathbf{v_2}(x|z_1)$.
4) $\frac{\mathbf{v_1}(x|z_1)}{\mathbf{v_1}(x|z_0)}$ and $\frac{\mathbf{v_2}(x|z_1)}{\mathbf{v_2}(x|z_0)}$ are increasing N-vectors.
5) $\frac{\mathbf{v_2}(x|z_0)}{\mathbf{v_1}(x|z_0)}$ and $\frac{\mathbf{v_2}(x|z_1)}{\mathbf{v_1}(x|z_1)}$ are increasing N-vectors.

Proposition 2. Given two vectors $\mathbf{x} = [x_1, x_2, \dots, x_N]$, $\mathbf{y} = [y_1, y_2, \dots, y_N]$, $\mathbf{x} \succ \mathbf{y}$, let $\mathbf{a} = [a_1, a_2, \dots, a_N]$ is an increasing N-vector, and there is at least $a_i \le 0$, we have:

$$\sum_{j=1}^N x_j a_j < \sum_{j=1}^N y_j a_j. \tag{16}$$

Proof: The proof is constructed by using the induction method. Of course, it is correct for $N = 1$ since $x_1 \ge y_1$ and a_1 is non-positive. Suppose that the inequality holds for $N = n - 1$,

$$\sum_{i=1}^{i=n-1} (x_i - y_i) a_i \le 0. \tag{17}$$

Next, we show that it holds for $N = n$. Indeed, since $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$, $x_n - y_n = \sum_{i=1}^{i=n-1} (y_i - x_i)$. Thus,

$$\begin{aligned}\sum_{i=1}^{i=n} (x_i - y_i) a_i &= \sum_{i=1}^{i=n-1} (x_i - y_i) a_i + (x_n - y_n) a_n \\ &= \sum_{i=1}^{i=n-1} (x_i - y_i) a_i + \sum_{i=1}^{i=n-1} (y_i - x_i) a_n = \sum_{i=1}^{i=n-1} (x_i - y_i)(a_i - a_n).\end{aligned} \tag{18}$$

Now, due to $\mathbf{a}$ is an increasing vector such that $a_i \leq a_{i+1}$, $[a_1 - a_n, a_2 - a_n, \ldots, a_{n-1} - a_n]$ is a negative increasing vector having $n-1$ elements. From $\sum_{i=1}^{n-1} x_i \geq \sum_{i=1}^{n-1} y_i$, we always can transform vector $\mathbf{x} = [x_1, x_2, \ldots, x_{n-2}, x_{n-1}]$ to $\mathbf{x}' = [x_1, x_2, \ldots, x_{n-2}, x'_{n-1}]$ by reducing x_{n-1} to x'_{n-1} where $x'_{n-1} = x_{n-1} - \delta$, $\delta \geq 0$, such that $\sum_{i=1}^{n-1} x'_i = \sum_{i=1}^{n-1} y_i$. From Definition 4, $\mathbf{x}' \succcurlyeq \mathbf{y}$. Thus, from assumption in (17), we have:

$$\begin{aligned}
&\sum_{i=1}^{i=n-1} (x_i - y_i)(a_i - a_n) \\
&= \sum_{i=1}^{i=n-2} (x_i - y_i)(a_i - a_n) + (x_{n-1} - y_{n-1})(a_{n-1} - a_n) \\
&= \sum_{i=1}^{i=n-2} (x_i - y_i)(a_i - a_n) + (x'_{n-1} + \delta - y_{n-1})(a_{n-1} - a_n) \quad (19) \\
&= \sum_{i=1}^{i=n-1} (x'_i - y_i)(a_i - a_n) + \delta(a_{n-1} - a_n) \quad (20) \\
&\leq \sum_{i=1}^{i=n-1} (x'_i - y_i)(a_i - a_n) \leq 0. \quad (21)
\end{aligned}$$

with (19) due to $x_{n-1} = x'_{n-1} + \delta$, (20) due to $\mathbf{x}' = [x_1, x_2, \ldots, x_{n-2}, x'_{n-1}]$, (21) due to $\delta \geq 0$, $a_{n-1} - a_n \leq 0$, and the assumption in (17) and $\mathbf{x}' \succcurlyeq \mathbf{y}$. Combining (18) and (21), $\sum_{i=1}^{i=n} (x_i - y_i) a_i \leq 0$ or the inequality holds for $N = n$. The proof is complete. ∎

Proposition 3. If the channel is dominated conditional distribution, for a given h (then given $\mathbf{v}(x|z_0)$ and $\mathbf{v}(x|z_1)$), $F(y) = 0$ has a unique finite solution, where:

$$F(y) = D(\mathbf{v}(x|y)||\mathbf{v}(x|z_0)) - D(\mathbf{v}(x|y)||\mathbf{v}(x|z_1)). \quad (22)$$

Proof:

$$\begin{aligned}
F(y) &= D(\mathbf{v}(x|y)||\mathbf{v}(x|z_0)) - D(\mathbf{v}(x|y)||\mathbf{v}(x|z_1)) \\
&= \sum_{i=1}^{N} p(x_i|y) \log \frac{p(x_i|z_1)}{p(x_i|z_0)}. \quad (23)
\end{aligned}$$

We show that $F(y)$ is a strictly increasing function i.e., for any $y_1 < y_2$, $F(y_1) < F(y_2)$. Indeed, let consider two real number y_1 and y_2 such that $y_1 < y_2$, from Proposition 1, $\mathbf{v}(x|y_1) \succ \mathbf{v}(x|y_2)$ and $\dfrac{\mathbf{v}(x|z_1)}{\mathbf{v}(x|z_0)}$ is an increasing N-vector. However, log is a monotonically increasing function, $\log \dfrac{\mathbf{v}(x|z_1)}{\mathbf{v}(x|z_0)}$ must be an increasing N-vector. Next, suppose that $\log \dfrac{\mathbf{v}(x|z_1)}{\mathbf{v}(x|z_0)}$ is a positive vector or $p(x_i|z_1) < p(x_i|z_0)$, $\forall i$. Thus, $1 = \sum_{i=1}^{N} p(x_i|z_1) < \sum_{i=1}^{N} p(x_i|z_0) = 1$. By contradiction method, existing at least a i such that $\log \dfrac{p(x_i|z_1)}{p(x_i|z_0)} < 0$.

Now, using Proposition 2 for $\mathbf{a} = \log \dfrac{\mathbf{v}(x|z_1)}{\mathbf{v}(x|z_0)}$, $\mathbf{x} = \mathbf{v}(x|y_1)$ and $\mathbf{y} = \mathbf{v}(x|y_2)$, $F(y_1) < F(y_2)$ if $y_1 < y_2$. Thus, $F(y)$ is a strictly increasing function. Next, we show that $F(y) = 0$ has at least one solution and it is finite. Indeed, from (13), $\dfrac{\phi_i(y)}{\phi_1(y)}$ is a monotonic strictly increasing function $\forall\ i \neq 1$, thus, if $y \to -\infty$ then $\dfrac{\phi_i(y)}{\phi_1(y)} \to 0$. Therefore, if $y \to -\infty$, $p(x_1|y) \to 1$ and $p(x_i|y) \to 0$, $i \neq 1$. Moreover, $\log \dfrac{\mathbf{v}(x|z_1)}{\mathbf{v}(x|z_0)}$ is an increasing vector which containing at least one negative entry, $\dfrac{p(x_1|z_1)}{p(x_1|z_0)} < 0$. Therefore, if $y \to +\infty$:

$$F(y) = \sum_{i=1}^{N} p(x_i|y) \log \frac{p(x_i|z_1)}{p(x_i|z_0)} = 1 \frac{p(x_1|z_1)}{p(x_1|z_0)} < 0. \quad (24)$$

A similar proof can be constructed to show that if $y \to +\infty$, $F(y) > 0$. From Rolle's theorem, $F(y) = 0$ has at least one solution and it must be finite. Finally, since $F(y)$ is a strictly increasing function, $F(y) = 0$ has a unique finite solution. ∎

1. Finite and unique $f(h)$**:** We are now ready to prove the first conclusion in Theorem 3 which states that for a given h, there exits a unique finite $f(h)$ satisfies (15). Indeed, due to the similarity of (22) and (15), the first conclusion in Theorem 3 is a directly subsequent of Proposition 3.

2. f(h) is a non-decreasing function: Next, we show that $f(h)$ is a non-decreasing function by contradiction. Indeed, suppose that existing $h_1 < h_2$ such that $f(h_1) = h'_1$, $f(h_2) = h'_2$ and $h'_1 > h'_2$.

Proposition 4. Consider a dominated conditional distribution channel and two real numbers h_1 and h_2 such that $h_2 - h_1 = \Delta$ where Δ is an arbitrary small positive number ($\Delta \to 0^+$), suppose that h_1 and h_2 induce $\mathbf{v_1}(x|z_0)$, $\mathbf{v_1}(x|z_1)$, $\mathbf{v_2}(x|z_0)$, $\mathbf{v_2}(x|z_1)$, respectively, and $f(h_1) = h'_1 > f(h_2) = h'_2$, then:

1) $\mathbf{v_1}(x|z_0) \succ \mathbf{v_2}(x|z_0) \succ \mathbf{v}(x|h'_2) \succ \mathbf{v}(x|h'_1)$.
2) $\mathbf{v}(x|h'_2) \succ \mathbf{v}(x|h'_1) \succ \mathbf{v_1}(x|z_1) \succ \mathbf{v_2}(x|z_1)$.

Now, from Proposition 1, $\dfrac{\mathbf{v_2}(x|z_0)}{\mathbf{v_1}(x|z_0)}$ is an increasing N-vector, from Proposition 4-1, $\mathbf{v_2}(x|z_0) \succ \mathbf{v}(x|h'_1)$. Thus, using Corollary 4 in [32] for $\mathbf{p} = \mathbf{v_2}(x|z_0)$, $\mathbf{q} = \mathbf{v_1}(x|z_0)$, and $\mathbf{r} = \mathbf{v}(x|h'_1)$, it is possible to show that:

$$D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_0)) > D(\mathbf{v}(x|h'_1)||\mathbf{v_2}(x|z_0)). \quad (25)$$

Symmetrically, one can verify that:

$$D(\mathbf{v}(x|h'_1)||\mathbf{v_2}(x|z_0)) > D(\mathbf{v}(x|h'_2)||\mathbf{v_2}(x|z_0)). \quad (26)$$

Combining (25) and (26), we have:

$$D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_0)) > D(\mathbf{v}(x|h'_2)||\mathbf{v_2}(x|z_0)). \quad (27)$$

Similarly, using Proposition 1, Proposition 4-2, and Corollary 4 in [32], we have:

$$D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_1)) < D(\mathbf{v}(x|h'_2)||\mathbf{v_2}(x|z_1)). \quad (28)$$

However, from the definition of $f(h)$ (Definition 3),

$$\begin{aligned}
D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_0)) &= D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_1)), \quad (29) \\
D(\mathbf{v}(x|h'_2)||\mathbf{v_2}(x|z_0)) &= D(\mathbf{v}(x|h'_2)||\mathbf{v_2}(x|z_1)). \quad (30)
\end{aligned}$$

From (27), (28) and (30),

$$D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_0)) > D(\mathbf{v}(x|h'_1)||\mathbf{v_1}(x|z_1)), \quad (31)$$

that contradicts to (29). By contradiction method, $h'_1 \leq h'_2$ or $f(h_1) \leq f(h_2)$ if $h_1 < h_2$. Thus, $f(h)$ is a non-decreasing function of h. The proof for Theorem 3 is complete.

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