Definition 0.1 *The bandwidth efficiency is defined as the ratio of the number of successfully transmitted data packets to that of the actual transmitted packets.*

By definition, the number of actual transmitted packets is always greater than or equal to the number of data packets due to the addition of either retransmitted packets. Thus, a scheme A is better than scheme B if it results in higher bandwidth efficiency. Furthermore, no scheme can have a bandwidth efficiency that is greater than 1.

1 Analysis of Transmission Techniques

In this section, we provide theoretical analysis for the ARQ, NC and RNC techniques for the single-hop wireless unicast network. First, for the sake of expository simplicity, we present the analysis for the case of one sender and two receivers.

1.1 Automatic Repeat reQuest (ARQ)

Using the ARQ scheme, the sender sends packets in sequence. If a packet loss occurs at some receiver, the receiver will send a NAK message to the sender to signal the sender to rebroadcast that lost packet. Our goal is to compute the bandwidth efficiency of this scheme, given the packet loss rates at different receivers. In the unicast scenario, each receiver wants to receive M distinct packets, so, the unicast bandwidth efficiency η_{UA} can be obtained by the following Proposition:

Proposition 1.1 *The bandwidth efficiency when using ARQ technique for two receivers with packet loss rates P_1 and P_2 is:*

$$\eta_{UA} = \frac{1}{\sum_{i=1}^{2} \frac{1}{2(1-P_i)}}. \tag{1}$$

Proof: The proof is straightforward. Since receiver R_i has a packet loss rate P_i; therefore, to transmit M packet to R_i successfully, the AP needs at least $\frac{M}{1-P_i}$ transmissions. Divide $2M$, the total number of useful data packets, by summation of the number of the required transmissions for all receivers, we obtain the proof.

1.2 Network Coding (NC)

We now analyze the network performance using network coding technique. Assume that R_1 wants to receive packet a_1 while R_2 wants to receive packet a_2. Clearly, if R_1 is willing to cache packet a_2 intended for R_2, and R_2 is willing to cache packet a_1 intended for R_1, then the two unicast sessions are now equivalent to a single broadcast session. Similarly, when there are K receivers that want to receive different packets, a receiver may want to cache everyone else's

Receiver	Packet								
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
R1	X	O	O	O	X	O	O	X	X
R2	O	X	O	X	O	O	X	X	O

Figure 1: An example of packet loss pattern.

data in order to use network coding for higher bandwidth efficiency. However, unlike the broadcast scenario with two receivers in which, a *combined* packet can be an XORed packet of any lost packets, in the unicast scenario, the *combined* packet must be a XOR combination of an even and an odd packet in order to be advantageous. This is because each receiver is only interested in receiving its own packets. For example, consider the loss patterns depicted in Fig. 1 where R_1 and R_2 want to receive odd and even packets, respectively. In this case, it is not advantageous to XOR packets a_7 and a_9 even though one successful transmission of this combined packet may allow R_1 to recover packet a_9 and R_2 to recover a_7. This is because R_2 does not want a_7, and a_7 will never be used in subsequent packet combining since R_1 already had packet a_7. Thus, the sender may as well send packet a_9 to avoid unnecessary coding. Using this unicast scheme, we have the following proposition:

Proposition 1.2 *The bandwidth efficiency when using network coding technique for two receivers with packet loss rates P_1 and P_2 is:*

$$\eta_{UN} = \frac{1}{1 + \frac{P_2}{2(1-P_2)} + \frac{P_1 P_2}{2(1-P_1)}}, \tag{2}$$

where $P_1 \leq P_2$ and M, the number of packets destined for each receiver, is sufficiently large.

Proof: Without loss of generality, assume that the receivers R_1 and R_2 want to receive the M odd and M even packets, respectively. The bandwidth gain of the network coding technique depends on how many pairs of lost packets among the two receivers that one can find in order to generate the combined packets. Furthermore, the average numbers of lost packets for R_1 and R_2 are MP_1 and MP_2, respectively. The retransmitted packets can be classified into two types: the combined and non-combined packets. As discussed previously, the sender only combines odd and even lost packets. One very important condition that an odd and an even packet can be combined together is the odd and even packets must be received correctly at R_2 and R_1, respectively. This implies that on average the number of packets one can pair up is $m = \min\{MP_1(1-P_2), MP_2(1-P_1)\} = MP_1(1-P_2)$ since $P_1 \leq P_2$ by assumption. As a result, there are $MP_1 - m$ and $MP_2 - m$ lost packets from R_1 and R_2 that need to be retransmitted as non-combined packets. Hence, the total number of transmissions needed to deliver M packets to each receiver successfully is

$$T = 2M + m.E[X_1] + (MP_1 - m).E[X_2] + (MP_2 - m).E[X_3] \tag{3}$$

where X_1, X_2 and X_3 are the random variables denoting the numbers of attempts before a successful transmission for the combined packets and non-combined packets for R_1 and R_2, respectively. X_2 and X_3 follow the geometric distribution, $E[X_2] = \frac{1}{1-P_1}$ and $E[X_3] = \frac{1}{1-P_2}$. Now, one can think of $E[X_1]$ as the expected number of transmissions per successful transmission in the NC broadcast scheme in which, the sender must transmit successfully a combined packet to both receivers. Therefore, we have

$$E[X_1] = \frac{1}{1 - max\{P_1, P_2\}} = \frac{1}{1 - P_2} \tag{4}$$

Substituting $E[X_1]$, $E[X_2]$ and $E[X_3]$ into (3) and dividing it by M we have the expected number of transmissions to successfully deliver two packets to R_1 and R_2

$$T_{UN} = 2 + \frac{P_1 P_2}{1 - P_1} + \frac{P_2}{1 - P_2} \tag{5}$$

Consequently, the bandwidth efficiency for NC unicast coding is

$$\eta_{UN} = \frac{1}{1 + \frac{P_1 P_2}{2(1-P_1)} + \frac{P_2}{2(1-P_2)}} \tag{6}$$

We can generalize the above result to K-receiver scenario.

Corollary 1.1 *The network bandwidth efficiency for K-receiver network and sufficiently large M using ARQ is:*

$$\eta_{UA} = \frac{1}{\frac{1}{K}\sum_{i=1}^{K} \frac{1}{1-P_i}}. \tag{7}$$

Theorem 1.1 *The network bandwidth efficiency for K-receiver network and sufficiently large M using NC is:*

$$\eta_{UN} = \frac{1}{1 + \frac{1}{K}\sum_{i=1}^{K} \frac{\prod_{j=i}^{K} P_j}{1-P_i}} \tag{8}$$

Proof: We prove it by induction. Without loss of generality we assume that $P_i \leq P_j$ if $i \leq j$, $\{i, j\} \in \{1, .., K\}$. First, let $K = 2$, we have

$$\begin{aligned} \eta_{UN} &= \frac{1}{1 + \frac{1}{2}\sum_{i=1}^{2} \frac{\prod_{j=i}^{2} P_j}{1-P_i}} \\ &= \frac{1}{1 + \frac{1}{2}\left(\frac{P_1 P_2}{1-P_1} + \frac{P_2}{1-P_2}\right)} \end{aligned} \tag{9}$$

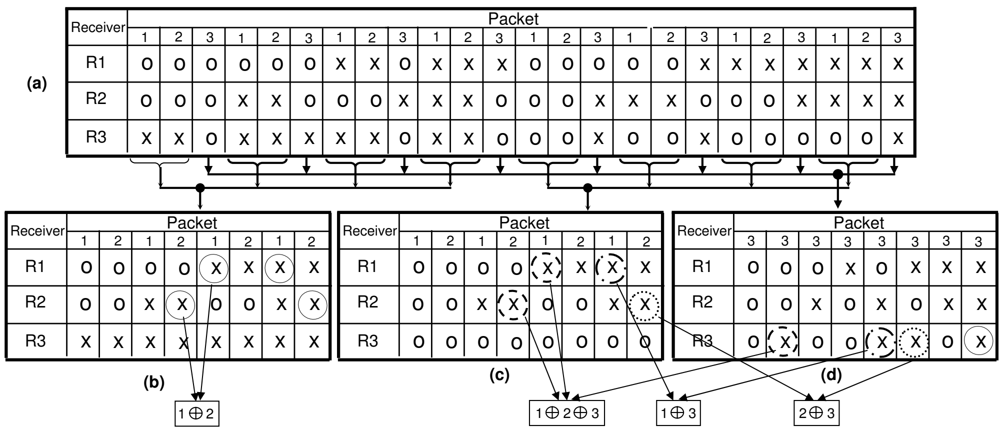

Figure 2: Error pattern for 3-receiver scenario. Packets numbered as 1, 2 and 3 denote time slots used for transmitting data for receiver R_1, R_2 and R_3, respectively. The circle patterns imply that the errors need to be retransmitted either in combined packets or non-combined packets.

The theorem holds for $K = 2$ since (9) was proved in Proposition (1.2).

We now prove that the theorem holds for $K = 3$. Fig. 2(a) and Fig. 2(b), (c) and (d), respectively, present the error pattern and its decomposed error patterns. Let us first consider the error pattern shown in Fig. 2(b) presenting a scenario in which the data destined to R_1 or R_2 and corrupted at R_3. Therefore, in the retransmission phase, the AP considers combining error packets, if possible, for R_1 and R_2 only and some non-combined packets will be transmitted alone. In other words, the AP uses the same combining strategy as that of the 2-receiver scenario. Therefore, the number of transmissions required to deliver the corrupted data which have error patterns as in Fig. 2(b) is

$$T_{UN}^{3}(1) = T_{UN}^{2} P_3, \tag{10}$$

where $T_{UN}^2 = \frac{MP_1P_2}{1-P_1} + \frac{MP_2}{1-P_2}$ denotes the number of retransmissions required to deliver the corrupted data for two receivers R_1 and R_2.

For the second and the third decompositions in Fig. 2(c) and (d), the AP combines corrupted data as $1 \oplus 2 \oplus 3$, $1 \oplus 3$ and $2 \oplus 3$. The number of available ingredient packets for each type of the coded packets is dominated by R_3, the receiver has the largest packet error probability. For example, in the combination for all receivers $1 \oplus 2 \oplus 3$, the number of available packets at R_1, R_2 and R_3 are $m_1 = MP_1(1-P_2)(1-P_3)$, $m_2 = MP_2(1-P_1)(1-P_3)$ and $m_3 = MP_3(1-P_1)(1-P_2)$, respectively. We can prove that $m_i \le m_j$ for $\forall i \le j$ based on the assumption $P_i \le P_j$. This implies that the ingredient packet constructing the coded packets for all receivers is dominated by the receiver with the highest packet error probability, m_3. The remaining which can not be combined will be transmitted to R_3 alone. The combinations are illustrated in

Fig. 2(c) and (d). Note that P_3 is the largest packet error probability; thus, the number of transmissions for delivering all the combined packets successfully depends only on P_3. Hence, the total number of retransmissions required to deliver all corrupted data for the second and the third decompositions is the number of transmissions required to deliver all error packets for R_3 only. That is

$$T_{UN}^{3}(2) = \frac{MP_3}{1-P_3}, \tag{11}$$

Adding up $3M$ time slots used for transmitting original packets with (10) and (11) we obtain the total number of transmissions needed to deliver all intended data. That is

$$\begin{aligned} T_{UN}^{3} &= 3M + T_{UN}^{3}(1) + T_{UN}^{3}(2) \\ &= 3M + \frac{MP_1P_2P_3}{1-P_1} + \frac{MP_2P_3}{1-P_2} + \frac{MP_3}{1-P_3} \end{aligned} \tag{12}$$

Divided $3M$, the total number of useful data packets, by T_{UN}^{3} we have the proof for the theorem for $K = 3$.

Now, suppose the theorem holds for $K = n-1$, $n \in N$, $n \geq 3$. This implies that the total number of required transmissions to deliver M packet for each receiver is

$$T_{UN}^{n-1} = (n-1)M + M\sum_{i=1}^{n-1} \frac{\prod_{j=i}^{n-1} P_j}{1-P_i} \tag{13}$$

We then prove that the theorem holds for $K = n$. Let T_{UN}^{n} denote the total number of transmissions needed to deliver M packets for each receiver. There are n receivers, therefore, the AP needs nM time slots to deliver the original packets for each receiver. In the retransmission phase, the AP considers using network coding to combine error packets. The error pattern is decomposed into three subsets S_1, S_2 and S_3. The set S_1 and S_2, respectively, present error patterns of packets destined to $\{R_1, ..., R_{n-1}\}$ while corrupted and succeeded at R_n; the set S_3 denotes the error patterns of packets destined to R_n at all receivers. Obviously, in the set S_1, the AP considers combining error packets for receivers $\{R_1, ..., R_{n-1}\}$ only since these packets are failure at R_n. Hence, the total number of time slots required for retransmitting error packets in set S_1 is the same as that of the number of time slots required for retransmitting error packets of the set $n-1$ receivers $\{R_1, ..., R_{n-1\}}$ only. That is

$$\begin{aligned} T_{UN}^{n}(1) &= M\left(\sum_{i=1}^{n-1} \frac{\prod_{j=i}^{n-1} P_j}{1-P_i}\right) P_n \\ &= M\sum_{i=1}^{n-1} \frac{\prod_{j=i}^{n} P_j}{1-P_i} \end{aligned} \tag{14}$$

An arbitrary error pattern of the set S_2 can be paired up with a pattern in S_3 to generate a coded packet. There are $2^{n-1}-1$ types a coded packet can be created. Note that in these combinations, every coded packet contains the information of data destined to R_n. Since $P_n = \max_{i\in\{1,\ldots,K\}}\{P_i\}$, therefore, the total number of time slots required to deliver all corrupted data for the errors in the set S_2 and S_3 equals to the number of time slots needed to deliver corrupted data for receiver R_n only. That is

$$T_{UN}^n(2) = \frac{MP_n}{1-P_n} \tag{15}$$

Adding up nM, the transmissions for original packets, with (14) and (15), the retransmissions for error packets, we obtain the total number of time slots needed to deliver all M packets for each receiver.

$$\begin{aligned} T_{UN}^n &= nM + M\sum_{i=1}^{n-1}\frac{\prod_{j=i}^n P_j}{1-P_i} + \frac{MP_n}{1-P_n} \\ &= nM + M\sum_{i=1}^{n}\frac{\prod_{j=i}^n P_j}{1-P_i} \end{aligned} \tag{16}$$

Divided nM by T_{UN}^n we have the proof for the theorem for $K = n$. By induction, the theorem holds for $\forall K \in \mathcal{N}$, $K \geq 2$. ■