Analysis of Adaptive Queueing Policies via Adiabatic Approach

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***Abstract*—We introduce an adiabatic framework for studying adaptive queuing policies. The adiabatic framework provides analytical tools for stability analysis of slowly changing systems that can be modeled as time-inhomogeneous reversible Markov chains. In particular, we consider queuing policies whose service rate is adaptively changed based on the estimated arrival rates that tend to vary with time. As a result, the packet distribution in the queue over time behaves like a time-inhomogeneous reversible Markov chain. Our results provide an upper bound on the time for an initial distribution of packets in the queue to converge to a stationary distribution corresponding to some pre-specified queueing policy. These results are useful for designing adaptive queueing policies when arrival rates are unknown, and may or may not change with time. Furthermore, our analysis is readily extended for any system that can be modeled as time-inhomogeneous reversible Markov chain. We provide simulations that confirms our theoretical results.**

I. Introduction

A Markov chain is a finite or countable state space random process where the probability of any particular future behavior of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behavior [1]. The theory of Markov chains is extensively used in queueing theory [2] and consequently in communication networks [3], among other fields. It is a well-known result that an irreducible, aperiodic Markov chain converges to a unique stationary distribution [4]. The time taken to converge to this distribution is of interest in applications like Monte Carlo Markov chain [4]. After a large number of steps the Markov chain must be "ϵ-close" according to some distance notion to its stationary distribution. This is the idea of Markov chain mixing and the time is quantified by mixing time [5]. In the case of time inhomogeneous Markov chains, the evolution must be slow enough such that the distribution is "ϵ-close" to the stationary distribution of the final transition matrix. This is the so-called adiabatic approach and this time is quantified by adiabatic time [6].

The adiabatic approach follows along the lines of the adiabatic evolution in quantum mechanics [7]. The quantum adiabatic theorem studies the evolution of a system from an initial Hamiltonian to a final Hamiltonian (operator corresponding to total energy of the system) and says that if the evolution is slow enough, the system will be "ϵ-close" in ℓ^2 norm to the ground state of the final Hamiltonian. We consider ℓ^1 norms in our study of Markov chains.

We study the convergence, in adiabatic sense, of time inhomogeneous Markov chains, where the time inhomogeneity can be due to an underlying nonstationary process or uncertainties in measurement of an underlying stationary process. In this paper, we apply the above convergence results to a Markovian queueing model where the time inhomogeneity arises due to the latter of the two conditions mentioned above.

A queue is typically defined by the arrival rate, service or departure rate, number of servers and buffer size. In a finite buffer size queue, we are often interested in maintaining a distribution which is more biased towards smaller queue lengths, since otherwise we will have high blocking probabilities. Such a stable queue is achieved by keeping the departure rate strictly above the arrival rate. However, it is physically impossible to have the departure rate arbitrarily higher than the arrival rate due to physical limitations such as the speed of the underlying router circuitry or the amount of power consumption. Furthermore, there might be other constraints on the departure rate due to considerations of the network as a whole. For example, it might not be best to always have a large departure rate that results in traffic bursts, and potentially causes congestion somewhere else in the network. Or in the case of using multiple queues of different kinds of traffic, each of which has to satisfy some pre-specified quality of service (QoS), sending packets from one queue will affect the other queues. As a result, controlling the packet sending rates depending on the types of traffic is desirable to allow for different flows to meet their QoS's requirements. In wireless networks, it is often preferable to maintain a certain departure rate and nothing more, due to power restrictions. Last but not least, in a network of multiple wireless nodes, the number of collisions is an increasing function of the traffic density. If every node sends its packets as fast as it can, then collisions will happen all the time. Hence it becomes necessary to monitor the departure rate and keep it at such a level to maintain a stable queue, and at the same time achieve the desired network objectives.

That said, we consider a queue in which packets arrive at a fixed but unknown rate and we use an estimate of this arrival rate to decide a queueing policy designed to ensure stable queues. In particular, we let the departure rate be always higher than the estimated arrival rate by some fixed multiplicative constant, anticipating that the estimate will be correct in the long-run. Since the estimated arrival rate is changed and is more accurate with time, the departure rate is also changed. As a result, the packets in the queue evolve according to a time inhomogeneous Markov chain dictated by this adaptive departure policy. We study the time required for the queue to reach this stationary distribution using the above outlined adiabatic approach, under suitable estimation and departure policies for the unknown arrival rate.

A. Related Work

The adiabatic theorem in quantum mechanics was first stated by Born and Fock [8]. A version of the adiabatic theorem [7] considers two Hamiltonians and the evolution of the system from the initial to final Hamiltonian. The theorem states that for sufficiently large time, the final state of the system and the ground state of the final Hamiltonian will be ϵ-close in ℓ^2 norm. The lower bound on time was found to be inversely proportional to the cube of the least spectral gap of the Hamiltonian over all time.

The adiabatic theorem for Markov chains was studied by Kovchegov [6] where the adiabatic evolution was studied for discrete time and continuous time Markov chains. The linear evolution of a time inhomogeneous Markov chain from an initial to a final probability transition matrix was studied and the adiabatic time was found to be proportional to the square of the mixing time or inversely proportional to the square of the spectral gap of the final transition matrix. This result was generalized to a more general adiabatic dynamics in [9].

B. Overview

- Adiabatic evolution of continuous time Markov chains with bounded generators where the changes happen at fixed time intervals.
- Upper bound on distance between the distribution of the Markov chain and the stationary distribution at large enough time.
- Application of the above upper bound to an M/M/1/K queueing model with unknown but constant arrival rate.
- Estimate of the arrival rate and corresponding dependent departure policy to ensure an eventually stable queue.
- Evolution of continuous time generator matrices determined by the changing, random departure rates based on the estimates of the arrival rate.
- The main result gives the sufficient time needed for the queue to converge to the eventual stable distribution.

Section II gives some of mathematical preliminaries including definitions and general results which we use in the analysis. In Section III, we look at the evolution of continuous time Markov chains whose generator matrices change at fixed time intervals and find an upper bound on the distance to a stationary distribution. In Section IV, the above result is applied to a queue and we consider a specific form of birth and death processes dictated by our estimation of the unknown arrival rate to the queue. The adiabatic time for this system is evaluated.We also compare the distance predicted by our upper bounds to the actual distance via simulations.

II. Preliminaries

Now we look at the main definitions and some results which will be useful in the rest of the paper. A vector x is a column vector whose ith entry is denoted by $x(i)$ and transpose denoted by x^T. A matrix P is a square matrix whose (i,j)th entry is denoted by P_{ij}.

- We use total variation distance to measure the distance between two probability distributions.
  Definition 1 (Total variation distance): For any two probability distributions ν and π on a finite state space Ω, we define
  $$\|\nu-\pi\|_{TV} = \frac{1}{2}\sum_{i\in\Omega}|\nu(i)-\pi(i)|.$$
  □
- *Definition 2:* Let π be a strictly positive probability distribution on a finite state space Ω and let $\ell^2(\frac{1}{\pi})$ be the real vector space $\mathbb{R}^r$ with the following norm,
  $$\|x\|_{\frac{1}{\pi}} \quad = \quad \left(\sum_{i\in\Omega}\frac{x(i)^2}{\pi(i)}\right)^{\frac{1}{2}}.$$
  □
- The following result is proved in Chapter 6, Theorem 3.2 of [4, p. 209].
  Proposition 3: For any two probability distributions ν and π (strictly positive),
  $$\|\nu-\pi\|_{TV} \le \frac{1}{2}\|\nu-\pi\|_{\frac{1}{\pi}}.$$
  □
- The following result is from Chapter 6, Theorem 3.3 of [4, p. 209].
  Proposition 4: Let P be a reversible[1]irreducible transition matrix on the finite state space Ω, with the stationary distribution π and second largest eigenvalue modulus $|\lambda_2(P)|$. Then for any probability distribution ν on Ω and for all $n\ge 1$
  $$\left\|\nu^T P^n - \pi^T\right\|_{\frac{1}{\pi}} \le |\lambda_2(P)|^n \left\|\nu-\pi\right\|_{\frac{1}{\pi}}.$$
  □
- For a continuous time Markov chain $\{X_t\}_{t\ge 0}$ on a finite state space Ω with a bounded generator matrix $Q=[Q_{ij}]_{i,j\in\Omega}$, and $q=\max_{i\in\Omega}\sum_{j:j\ne i}Q_{ij}$, the upper bound on the departure rates of all states, uniformization [10] gives transition probabilities to be
  $$P(t)=\sum_{n=0}^{\infty}e^{-qt}\frac{(qt)^n}{n!}P^n = e^{Qt}, \tag{1}$$
  where the matrix $P=I+\frac{1}{q}Q$. The matrix $P(t)$ denotes the transition probabilities in time t.
- The following result is used to bound the distance of the continuous chain in terms of a discrete one and is based

[1]Transition probabilities of a reversible Markov chain satisfy $p_{ij}\pi_i = p_{ji}\pi_i$

Theorem 1: For the time inhomogeneous Markov chain generated by the matrices $\{Q_i\}_{i=0}^n = \{q_i(P_i - I)\}_{i=0}^n$ from time 0 to $n\Delta t$,

$$\begin{aligned} \|\nu_n - \pi_n\|_{TV} &\leq \frac{1}{2}\|\nu_{n_0} - \pi_{n_0}\|_{\frac{1}{\pi_{n_0}}} \prod_{i=n_0}^{n-1} e^{-q_i(1-|\lambda_2(P_i)|)\Delta t} \sqrt{\max_{k\in\Omega} \frac{\pi_i(k)}{\pi_{i+1}(k)}} \\ &+ \frac{1}{2}\sum_{i=n_0}^{n-1} \left[\|\pi_i - \pi_{i+1}\|_{\frac{1}{\pi_{i+1}}} \prod_{j=i+1}^{n-1} e^{-q_j(1-|\lambda_2(P_j)|)\Delta t} \sqrt{\max_{k\in\Omega} \frac{\pi_j(k)}{\pi_{j+1}(k)}} \right], \end{aligned}$$

where ν_n is the distribution at time $n\Delta t$, ν_{n_0} is the distribution at time $n_0\Delta t$ for $n_0 < n$ and $\{P_i\}_i$ are irreducible and reversible with stationary distribution π_i and second largest eigenvalue modulus $|\lambda_2(P_i)|$. $\square$

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on [4, p. 364]. The proof follows along similar lines and can be found in [11].

Proposition 5: For a continuous time Markov chain on a finite state space Ω with generator matrix $Q = q(P - I)$ with reversible, irreducible P and stationary distribution π, for any probability distribution ν on Ω,

$$\left\|\nu^T e^{Qt} - \pi^T\right\|_{\frac{1}{\pi}} \leq \|\nu - \pi\|_{\frac{1}{\pi}} e^{-q(1-|\lambda_2(P)|)t},$$

where $|\lambda_2(P)|$ is the second largest eigenvalue modulus of P. $\square$

- Mixing time of a Markov chain measures the time needed for the Markov chain to converge to its stationary distribution.

  Definition 6: For a continuous time Markov chain $P(t)$, with stationary distribution π, given an $\epsilon > 0$, the mixing time $t_{mix}(\epsilon)$ is defined as

  $$\begin{aligned} t_{mix}(\epsilon) &= \inf\left\{t : \|\nu^T P(t) - \pi^T\|_{TV} \leq \epsilon, \text{for all} \right. \\ &\qquad \left. \text{probability distributions } \nu\right\}. \end{aligned}$$

  $\square$

- To define adiabatic time, consider a linear evolution [6] of generator matrices described by

  $$Q\left(\frac{t}{T}\right) = \left(1 - \frac{t}{T}\right) Q_{initial} + \frac{t}{T} Q_{final},$$

  for $T > 0$, $0 \leq t \leq T$ and bounded generators $Q_{initial}$ and Q_{final}. If π_f is the unique stationary distribution for Q_{final}, adiabatic time measures the time needed for the chain to converge to π_f.

  Definition 7: Given the above transitions generating a continuous time Markov chain $P(0,T)$ and $\epsilon > 0$, adiabatic time is defined as

  $$\begin{aligned} T_\epsilon &= \inf\left\{T : \|\nu^T P(0,T) - \pi_f^T\|_{TV} \leq \epsilon, \text{for all} \right. \\ &\qquad \left. \text{probability distributions } \nu\right\}. \end{aligned}$$

  We will look at a different evolution of generator matrices in the next section. $\square$

III. Adiabatic Framework

In this section we define an evolution of continuous time Markov generator matrices and look at the convergence in Definition (7) in terms of this new evolution. Consider the following evolution: time is divided into slots of size Δt and the generator matrix changes at these intervals. The bounded generator matrix Q_i determines the transition probabilities in the time interval $(i\Delta t, (i+1)\Delta t]$. The method of uniformization gives the corresponding transition probability matrix $P(i\Delta t, (i+1)\Delta t)$ as in (1). The upper bound on departure rates over all states is q_i for each Q_i. Therefore,

$$P(i\Delta t, (i+1)\Delta t) = e^{Q_i\Delta t} = e^{q_i(P_i - I)\Delta t},$$

where $P(t_1, t_2)$ denotes the matrix of transition probabilities from time t_1 to t_2. Let the matrix P_i be irreducible and reversible with stationary distribution π_i and second largest eigenvalue modulus $|\lambda_2(P_i)|$. Note that this evolution can be the result of any kind of time inhomogeneity in the system resulting in a changing Q_i which is updated at fixed intervals of time. As noted before, the time inhomogeneity can be due to the nature of the underlying process or due to uncertainties in measurements of parameters and Theorem 1 captures either kind.

Let ν_n be the distribution of the chain at time $n\Delta t$. We are interested in the distance between ν_n and the stationary distribution π_n corresponding to matrix P_n at time $n\Delta t$. Theorem 1 gives an upper bound on the distance at time $n\Delta t$ in terms of the distance at $n_0\Delta t$ for $n_0 < n$. The proof follows by starting from the $\frac{1}{\pi_n}$ norm and then applying the triangle inequality repeatedly. Propositions 3and 5 give the required upper bounds.

IV. Application to Queueing Model

In this section we apply the above defined adiabatic evolution model to a queueing process. In particular, we consider time inhomogeneity due to uncertainty in a parameter. Consider an M/M/1/K queue with unknown packet arrival rate λ per unit time. We estimate λ at time $i\Delta t$ denoted by $\hat{\lambda}_i$ and decide packet departure rate, $\mu_i = f(\hat{\lambda}_i)$ based on this estimate.

Definition 8: Queueing policy is defined as the sequence $\{\hat{\lambda}_i, \mu_i = f(\hat{\lambda}_i)\}_{i\geq 1}$ where $f : \mathbb{R}_+ \to \mathbb{R}_+$ and μ_i is applied for time from $(i\Delta t, (i+1)\Delta t]$. $\square$

Theorem 2 (Main result): Given $0 < \epsilon < 1$, $0 < \gamma < 1$ and λ, the unknown arrival rate, the queueing policy in Definition (9) with $\delta > 0$ for an M/M/1/K queue has

$$T_{\epsilon,\gamma} \leq \frac{2\log\frac{2}{\gamma_1}}{\lambda(\epsilon_0^2-\epsilon_0^3)} + \frac{\log\Big(2\big[(1+\epsilon_0)(1+\delta)\big]^{\frac{K}{2}+1}\Big) - \log(\epsilon\delta)}{\frac{1}{2\Delta t}\log\big(\frac{\left[(1-\epsilon_0)(1+\delta)\right]^{K+1}-1}{\left[(1-\epsilon_0+\epsilon_1)(1+\delta)\right]^{K+1}-1}\big) + \lambda(\sqrt{(1+\delta)(1-\epsilon_0)}-1)^2}, \tag{3}$$

where ϵ_0 satisfies

$$\begin{aligned} e^{-\lambda\Delta t(\sqrt{(1+\delta)(1-\epsilon_0)}-1)^2}\sqrt{\frac{\big[(1-\epsilon_0+\epsilon_1)(1+\delta)\big]^{K+1}-1}{\big[(1-\epsilon_0)(1+\delta)\big]^{K+1}-1}} &< 1, && (4)\\ \frac{\frac{\sqrt{(1-\epsilon_0)(1+\delta)}\epsilon_1}{|(1-\epsilon_0)^2(1+\delta)-(1-\epsilon_0+\epsilon_1)|}}{1-e^{-\lambda\Delta t(\sqrt{(1+\delta)(1-\epsilon_0)}-1)^2}\sqrt{\frac{\left[(1-\epsilon_0+\epsilon_1)(1+\delta)\right]^{K+1}-1}{\left[(1-\epsilon_0)(1+\delta)\right]^{K+1}-1}}} &\leq \frac{\epsilon}{2}, && (5)\\ \epsilon_0 &\leq \frac{\delta\epsilon}{(1+\delta)(1+\epsilon)}, && (6)\end{aligned}$$

and $\epsilon_1 = \frac{\lambda\Delta t(\epsilon_0^2-\epsilon_0^3)}{2\log\frac{2}{\gamma_1}}(\frac{1}{\sqrt{\lambda\Delta t\gamma_2}}+\epsilon_0), 0 < \gamma_1 < \gamma, \gamma_2 = \gamma - \gamma_1$. □

This decides the following generator matrix from $(i\Delta t, (i+1)\Delta t]$:

$$Q_i = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu_i & -(\mu_i+\lambda) & \lambda & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & 0 & \mu_i & -(\mu_i+\lambda) & \lambda \\ \dots & 0 & 0 & \mu_i & -\mu_i \end{pmatrix}. \tag{2}$$

The corresponding transition probability matrix $P(i\Delta t, (i+1)\Delta t)$ is obtained as in (1). The upper bound on departure rates over all states is $\lambda + \mu_i$. Therefore,

$$P(i\Delta t, (i+1)\Delta t) = e^{Q_i\Delta t} = e^{(\lambda+\mu_i)(P_i-I)\Delta t},$$

where the matrix, P_i is

$$P_i = \begin{pmatrix} 1-\beta_i & \beta_i & 0 & 0 & \dots \\ 1-\beta_i & 0 & \beta_i & 0 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & 0 & 1-\beta_i & 0 & \beta_i \\ \dots & 0 & 0 & 1-\beta_i & \beta_i \end{pmatrix}, \tag{7}$$

where $\beta_i = \frac{\lambda}{\mu_i+\lambda}$. The P_i's are reversible and irreducible with stationary distribution given by

$$\pi_i = \frac{1}{\sum_{r=0}^{K}\rho_i^r}\left[1, \rho_i, \rho_i^2, \dots, \rho_i^{K-1}, \rho_i^K\right]^T,$$

where $\rho_i = \frac{\beta_i}{1-\beta_i} = \frac{\lambda}{\mu_i}$. The second largest eigenvalue modulus is given by $|\lambda_2(P_i)| = 2\frac{\sqrt{\rho_i}}{(1+\rho_i)}\cos(\frac{\pi}{K+1})$ (proved in [12]). Also, the matrix Q_0 could be decided from a random departure rate and corresponds to the transition probability matrix $P(0, \Delta t)$.

A. Performance of an Adaptive Queueing Policy

Now we look at a specific queueing policy determined by the time average of number of packets arrived.

Definition 9:

$$\begin{aligned} \hat{\lambda}_i &= \frac{1}{i\Delta t}\sum_{k=1}^{i} X_k, \\ \mu_i &= f(\hat{\lambda}_i) = (1+\delta)\hat{\lambda}_i, \end{aligned}$$

where $X_k \sim$ Poisson$(\lambda\Delta t)$ is the number of packets in the kth slot of duration Δt and $\delta > 0$ is a constant. □

This particular queueing policy ensures that the departure rate is always higher than the estimated arrival rate and since the estimated arrival rate itself must approach the actual one, should ensure a stable queue.

With this queueing policy, we have the adiabatic evolution generated by the matrices Q_i in (2). The corresponding P_i's are given by (7). The ratio $\rho_i = \frac{\lambda}{\mu_i} = \frac{\lambda}{(1+\delta)\hat{\lambda}_i}$. With full knowledge of the arrival rate, the above ratio becomes $\rho = \frac{1}{1+\delta}$ and let the corresponding matrix P has stationary distribution π. We redefine adibatic time in this setting to

Definition 10: Given the above transitions generating a continuous time Markov chain $P(0, n\Delta t)$, $\epsilon > 0$ and $\gamma < 1$, adiabatic time is defined as

$$\begin{aligned} T_{\epsilon,\gamma} \;=\; & \Delta t \cdot \inf\big\{n : \Pr\{\|\nu^T P(0, n\Delta t) - \pi^T\|_{TV} < \epsilon\} \\ & > 1-\gamma \text{ , for all probability distributions } \nu\big\}. \end{aligned}$$

where π is the stationary distribution corresponding to the queueing policy $\{\lambda, (1+\delta)\lambda\}$. □

Theorem 2 gives a sufficient condition on the time we must wait before the distribution of the queue length converges to the desired stationary distribution π. Note that π is decided by δ and can be designed to give a stable stationary distribution.

Hence, for given small ϵ and γ, the theorem gives the sufficient amount of time to converge to a stable distribution within ϵ with high probability of $1-\gamma$. The choice of ϵ_0 to be the largest which satisfies all three conditions will give the lowest lower bound in the theorem. At large enough time the estimated arrival rate must approach the actual arrival rate and the difference can be bounded by ϵ_0 with a high probability. Furthermore two consecutive estimates, can differ only by a maximum of ϵ_1. The proof sketch of this theorem is in the appendix.

B. Simulations

Now we look at the distance of the distribution of the queue from a stable distribution with increasing time. This distance must be small for large enough time from the discussions above. Here we look at distance as a function of time for a single sample path to verify whether the distance at the adiabatic time predicted is indeed lower than ϵ.

Simulation Setup 1: $\lambda = 10, \delta = 0.1, K = 100, \Delta t = 0.5, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$.

Theorem 2 predicts $n_0 = 174391$ and $n = 175438$ for $\epsilon_0 = 0.003$ (the highest which satisfies all the conditions in Theorem 2) or adiabatic time for $\epsilon = 0.1, \gamma = 0.05$, $T_{0.1,0.05} = 87719$. Figure 1 shows the results of a simulation which measures the distance at each time slot vs. the distance upper bound given by triangle inequality in Theorem 1. The triangle inequality upper bound very closely follows the actual distance and both are much smaller than the distance upper bound predicted by (15) and (17), which is 0.056. Note that this is only a sample path. This indicates that even though Theorem 2 gives $T_{0.1,0.05} = 87719$ as a sufficient condition, the distance of $\epsilon = 0.1$ is achieved much before. For example, Figure 2 shows the simulation results for $T = 5000$ or $n = 10000$. (The upper bound starts from $n_0 = 500$.) This was also found to be true for all 100 sample paths tested.

Notice the initial increase in the upper bound due to Theorem 1 in both Figures 1 and 2. This is because of the second term of the bound which adds terms with time. After a sufficient number of time slots, however, the decrease in the first term overshadows the increase in the second and the overall bound starts coming down.

Now we look at the effects of increasing δ which decides how fast the departures are compared to the arrivals.

Simulation Setup 2: $\lambda = 10, K = 100, \Delta t = 0.5, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$ with different $\delta = 0.1 : 0.1 : 0.9$.

Note that increase in δ corresponds to emptying out the queue faster or in other words, the stable distribution should be approached faster. Or, at the same time, the distance decreases with increasing δ. The distances at $T = 2500$ averaged over 100 sample paths are shown in Figure 3 which confirms this.

Another parameter of interest is the sampling time Δt which decides how frequently the estimates are done.

Simulation Setup 3: $\lambda = 10, K = 100, \delta = 0.1, \epsilon = 0.1, \gamma = 0.05, \gamma_1 = 0.04$ with different $\Delta t = 1, 5, 10, 20, 25, 50$.

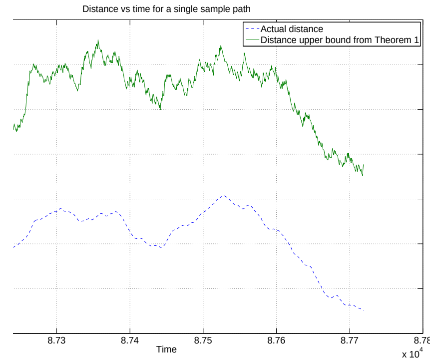

Figure 1. Comparison of distance upper bound in Theorem 1 with actual distance

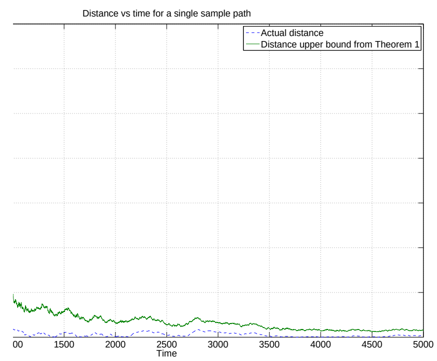

Figure 2. Comparison of distance upper bound with actual distance at lesser time than predicted by Theorem 2

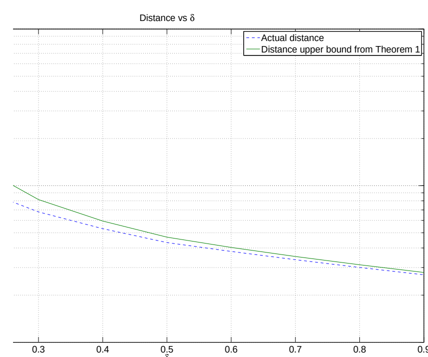

Figure 3. Comparison of distance for different δ

The increase in Δt corresponds to applying the information gained from estimates less frequently and must result in an increased distance at the same time. The distances at time $T = 100$ averaged over 100 sample paths are shown in Figure 4 which shows that this is true. This shows the benefits of updates which are more frequent and hence the system following a smoother path towards the stable distribution rather than attempting to reach there at one go.

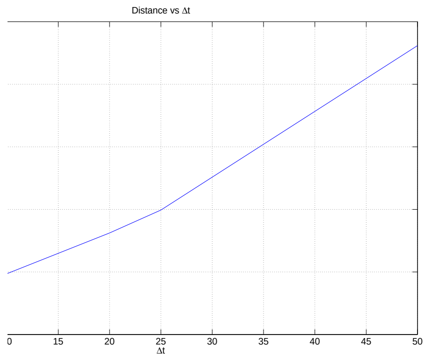

Figure 4. Comparison of distance for different Δt.

V. Conclusion

We considered a time inhomogeneous Markov chain with its evolution governed by generator matrices which change at fixed time intervals. We followed the adiabatic approach to study this chain and bounded the distance to a stationary distribution. This is a general theorem which is applicable to evolutions of the above kind.

We used this model of time inhomogeneous Markov chains to characterize a birth-death chain in which the time inhomogeneity is due to uncertainty in parameters. We considered a Markovian queue with an adaptive queueing policy and derived sufficient conditions on time after which the distribution of the queue can be considered to be stable with high probability.

Appendix A
Proof Sketch of Theorem 2

To prove Theorem 2, we bound the terms in Theorem 1 as in the following lemma.

Lemma 3: For $0 < \epsilon_0 < 1$ and $0 < \gamma_1 < 1$, there exists $n_0 = \frac{2\log\frac{2}{\gamma_1}}{\lambda\Delta t(\epsilon_0^2 - \epsilon_0^3)}$ such that

- $\forall i \geq n_0$, with probability at least $1-\gamma_1$

$$\begin{aligned} |\hat{\lambda}_i - \lambda| &\leq \lambda\epsilon_0, &(8)\\ e^{-(\lambda+\mu_i)(1-|\lambda_2(P_i)|)\Delta t} &< A, &(9)\\ \|\pi_i - \pi\|_{TV} &< \frac{1}{2}\frac{\epsilon_0(1+\delta)}{\delta - \epsilon_0(1+\delta)}, &(10)\end{aligned}$$

  where $A = e^{-\lambda\Delta t(\sqrt{(1+\delta)(1-\epsilon_0)}-1)^2}$.

- For $\epsilon_1 = \frac{1}{n_0}(\frac{1}{\sqrt{\lambda\Delta t\gamma_2}} + \epsilon_0)$ and $0 < \gamma_2 < 1$ and $\forall i \geq n_0$, with probability at least $1-\gamma = 1-\gamma_1-\gamma_2$

$$\begin{aligned} |\hat{\lambda}_{i+1} - \hat{\lambda}_i| &< \lambda\epsilon_1, &(11)\\ \sqrt{\max_{k\in\{0,1,\dots,K\}}\frac{\pi_i(k)}{\pi_{i+1}(k)}} &< B, &(12)\\ \|\pi_i - \pi_{i+1}\|_{\frac{1}{\pi_{i+1}}} &< C, &(13)\end{aligned}$$

  where $B = \sqrt{\frac{\left[(1-\epsilon_0+\epsilon_1)(1+\delta)\right]^{K+1}-1}{\left[(1-\epsilon_0)(1+\delta)\right]^{K+1}-1}}$ and $C = \frac{\sqrt{(1-\epsilon_0)(1+\delta)\epsilon_1}}{|(1-\epsilon_0)^2(1+\delta)-(1-\epsilon_0+\epsilon_1)|}$.

- With probability at least $1-\gamma_1$,

$$\|\nu_{n_0} - \pi_{n_0}\|_{\frac{1}{\pi_{n_0}}} < \frac{\left[(1+\epsilon_0)(1+\delta)\right]^{\frac{K}{2}+1}}{\delta}. \quad (14)$$

□

The proof is omitted for brevity and can be found in [11].

Rewriting Theorem 1 for the queueing model we have described and applying Lemma 3, with probability at least $1-\gamma_1-\gamma_2 = 1-\gamma$,

$$\begin{aligned} \|\nu_n - \pi_n\|_{TV} &< \frac{1}{2}\frac{\left[(1+\epsilon_0)(1+\delta)\right]^{\frac{K}{2}+1}}{\delta}[C.D]^{n-n_0}\\ &\quad + \frac{1}{2}\frac{C}{1-A.B} \leq \frac{\epsilon}{2}, \quad (15)\end{aligned}$$

where (15) holds if ϵ_0 is chosen to satisfy (4).

Now the distance between ν_n and π which is the stationary distribution corresponding to the matrix P with full knowledge of λ is given by

$$\|\nu_n - \pi\|_{TV} \leq \|\nu_n - \pi_n\|_{TV} + \|\pi_n - \pi\|_{TV}, \quad (16)$$

using triangle inequality. Our aim is to find the n such that $\|\nu_n - \pi\|_{TV} < \epsilon$. (15) holds if both terms are $\leq \frac{\epsilon}{4}$. For the first term we get,

$$n - n_0 \geq \frac{\log\left(\frac{2\left[(1+\epsilon_0)(1+\delta)\right]^{\frac{K}{2}+1}}{\epsilon\delta}\right)}{\log(\frac{1}{AB})},$$

which gives a condition on n. Adding with n_0 and multiplying by Δt gives the adiabatic time as defined in Definition (10). The second term of (15) $\leq \frac{\epsilon}{4}$ gives (5).

From (16) and (10),

$$\|\pi_n - \pi\|_{TV} < \frac{1}{2}\frac{\epsilon_0(1+\delta)}{\delta-\epsilon_0(1+\delta)} \leq \frac{\epsilon}{2}, \quad (17)$$

which gives (6). This completes the proof of Theorem 2.

References

[1] S. Karlin and H. M. Taylor, *A first course in stochastic processes*, 2nd ed. New York: Academic Press, 1975.
[2] L. Kleinrock, *Queueing systems, volume 1: Theory*. New York: John Wiley & Sons, 1975.
[3] ——, *Queueing systems, volume 2: Computer applications*. New York: John Wiley & Sons, 1976.
[4] P. Brémaud, *Markov chains: Gibbs fields, Monte Carlo simulation, and queues*. New York: Springer-Verlag, 1999.
[5] D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov chains and mixing times*, 1st ed. American Mathematical Society, December 2008.
[6] Y. Kovchegov, "A note on adiabatic theorem for Markov chains," *Statistics and Probability Letters*, vol. 80, pp. 186–190, 2010.
[7] A. Messiah, *Quantum mechanics, volume 2*, 1st ed. New York: John Wiley & Sons, 1962.
[8] M. Born and V. Fock, "Beweis des adiabatensatzes," *Zeitschrift Für Physik A Hadrons and Nuclei*, vol. 51, no. 3-4, pp. 165–180, 1928.
[9] K. Bradford and Y. Kovchegov, "Adiabatic times for Markov chains and applications," *Journal of Statistical Physics*, vol. 143, pp. 955–969, 2011.
[10] S. M. Ross, *Introduction to probability models*, 10th ed. Academic Press, 2010.
[11] L. Zacharias, "An adiabatic approach to analysis of time inhomogeneous Markov chains: a queueing policy application," Master's thesis, Oregon State University, Corvallis, USA, September 2011.
[12] W. Feller, *An introduction to probability theory and its applications, volume 1*, 2nd ed. New York: John Wiley & Sons, November 1959.