This is the author's version of an article that has been published in this journal. Changes were made to this version by the publisher prior to publication.
The final version of record is available at http://dx.doi.org/10.1109/TVT.2020.3003560

On Bounds and Closed-Form Expressions for Capacities of Discrete Memoryless Channels with Invertible Positive Matrices

Thuan Nguyen
School of Electrical and
Computer Engineering
Oregon State University
Corvallis, OR, 97331
Email: nguyeth9@oregonstate.edu

Thinh Nguyen
School of Electrical and
Computer Engineering
Oregon State University
Corvallis, 97331
Email: thinhq@eecs.oregonstate.edu

***Abstract*—While capacities of discrete memoryless channels are well studied, it is still not possible to obtain a closed-form expression for the capacity of an arbitrary discrete memoryless channel (DMC). In this paper, we study a class of DMCs whose channel matrix is an invertible positive matrix. This class of channel matrices can be used to model many real-world settings. Next, an elementary technique based on Karush-Kuhn-Tucker (KKT) conditions is used to obtain (1) a good upper bound of a discrete memoryless channel having an invertible positive channel matrix and (2) a closed-form expression for the capacity if the channel matrix satisfies certain conditions related to its singular value and its Gershgorin's disk.**

***Index Terms*—Wireless Communication, Convex Optimization, Channel Capacity, Mutual Information.**

I. Introduction

Discrete memoryless channels (DMC) play a critical role in the early development of information theory and its applications. DMCs are especially useful for studying many well-known modulation/demodulation schemes (e.g., PSK and QAM) in which the continuous inputs and outputs of a channel are quantized into discrete symbols. Thus, there exists a rich literature on the capacities of DMCs [1], [2], [3], [4], [5], [6], [7]. In particular, capacities of many well-known channels such as (weakly) symmetric channels can be written in elementary formulas [1]. However, it is often not possible to express the capacity of an arbitrary DMC in a closed-form expression [1]. Recently, several papers have been able to obtain closed-form expressions for a small class of DMCs with small alphabets. For example, Martin et al. established closed-form expression for a general binary channel [8]. Liang showed that the capacity of channels with two inputs and three outputs can be expressed as an infinite series [9]. Paul Cotae et al. found the capacity of two input and two output channels in term of the eigenvalues of the channel matrices [10]. In [11], the authors used geometric programming to construct a simple closed-form expression for the upper bound of the capacity of an arbitrary DMC. It is worth noting that the approach in [11] based on elementary Lagrange functions is similar to our approach in this paper. On the other hand, the problem of finding the capacity of a discrete memoryless channel can be formulated as a convex optimization problem [12], [13]. Thus, efficient algorithmic solutions exist. There are also iterative algorithms such as Arimoto-Blahut algorithm [2], [3] and other variants for computing channel capacities [14], [15], [16], [17], [18]. Even though there exist efficient algorithms for finding the capacity of an arbitrary DMC, there are a number of reasons why channel capacity or bounds expressed in closed-form expression can be very useful. These include (1) formulas can often provide a good intuition about the relationship between the capacity and different channel parameters, (2) formulas offer a faster way to determine the capacity than that of algorithms, and (3) formulas are useful for analytical derivations where closed-form expression of the capacity is needed in the intermediate steps. Moreover, the channel capacity or bounds expressed in closed-form expression might be particularly useful for channels having large alphabet sizes since the well-known Arimoto-Blahut algorithm already provides the capacity values fairly quickly for channels with small alphabet sizes. In fact, our work is motivated by our current work on a prototype of a Free Space Optical communication system called WiFO [19]. WiFO's transceiver is capable of adjusting transmitting and receiving parameters for power and coverage optimization. The result is that the channel matrix can be changed dynamically. For a given channel matrix, we want to know the closed-form expression of the channel capacity so that a trade-off among power consumption, coverage, and capacity can be optimized quickly.

To that end, in this paper we investigate the closed-form expressions for the capacities and their upper bounds of an important class of DMCs whose channel matrices are invertible positive matrices. An invertible positive matrix is a square matrix whose entries are strictly greater than zero and invertible. There are a number of reasons for using an invertible positive matrix to model many communication channels in real-world settings. First, in most digital communication systems, the transmitter sends a set of transmitted symbols (inputs) and the receiver aims to decode the received signals into one of the transmitted symbols (outputs). Consequently, the channel matrix is a square matrix consisting of the same number of inputs and outputs. Second, since it is physically impossible to design a communication channel without error, the assumption on the entries in the channel matrix to be strictly greater than zero is reasonable. In the case when an entry is truly zero, it is always possible to approximate the zero with a small positive number. Third, for a $n \times n$ matrix, if the entries are drawn uniformly from a real set (or more precisely in (0,1) and the rows form a valid conditional pmf), then it can be shown that the probability of the matrix being invertible is approaching 1 with increasing n. Thus, invertible matrices are arguably useful to model many communication channels in real-world settings.

Building on the work in [7], our contributions include: (1) we describe an elementary technique based on the theory of convex optimization, to find the closed-form expression for a good upper bound on capacities of discrete memoryless channels with positive invertible channel matrix, and (2) we find the optimality conditions of the channel matrix for which the upper bound is precisely the capacity. We refine the optimality conditions in [7] and provide additional easy-to-use conditions for obtaining closed-form expression for capacities. In particular, the optimality conditions establish a relationship between the singular value and the Gershgorin's disk of the channel matrix. Intuitively, this optimality condition of a channel matrix corresponds to the channel matrix belonging to a subclass of strictly diagonally dominant matrices. Since strictly diagonally dominant matrices represent reliable channels (to be discussed), our results could be useful since most communication systems are designed to achieve a certain level of reliability. Furthermore, our results extend the class of channel matrices, especially the symmetric and weakly symmetric matrices whose channel capacities can be found in closed-form expressions.

II. Preliminaries

In this section, we provide definitions together with elementary results that will aid our discussions. In particular, we will discuss (1) the optimality KKT conditions and (2) linear algebra results which we use to derive the closed-form expressions for both the capacity upper bound and exact capacity.

A. Convex Optimization and KKT Conditions

A DMC is characterized by a random variable $X \in \{x_1, x_2, \dots, x_m\}$ for the inputs, a random variable $Y \in \{y_1, y_2, \dots, y_n\}$ for the outputs, and a channel matrix $A \in \mathbf{R}^{m\times n}$. In this paper, we consider DMCs with equal number of inputs and outputs n, thus $A \in \mathbf{R}^{n\times n}$. The matrix entry A_{ij} represents the conditional probability that given x_i is transmitted, y_j is received. Let $p = (p_1, p_2, \dots, p_n)^T$ be the input probability mass vector (pmf) of X, where p_i denotes the probability of x_i to be transmitted, then the pmf of Y is $q = (q_1, q_2, \dots, q_n)^T = A^T p$ and A^T denotes the transpose of A. The mutual information between X and Y is:

$$I(X;Y) = H(Y) - H(Y|X), \tag{1}$$

where

$$H(Y) = -\sum_{j=1}^{n} q_j \log q_j \tag{2}$$

$$H(Y|X) = -\sum_{i=1}^{n}\sum_{j=1}^{n} p_i A_{ij} \log A_{ij}. \tag{3}$$

The mutual information function can be written as:

$$I(X;Y) = -\sum_{j=1}^{n} (A^T p)_j \log (A^T p)_j + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i A_{ij} \log A_{ij}, \tag{4}$$

where $(A^T p)_j$ denotes the j^{th} component of the vector $q = (A^T p)$. The capacity C associated with a channel matrix A is the theoretical maximum rate at which information can be transmitted over the channel without the error [5], [20], [21]. It is obtained using the optimal pmf p^* such that $I(X;Y)$ is maximized. For a given channel matrix A, $I(X;Y)$ is a concave function of p [1]. Therefore, maximizing $I(X;Y)$ is equivalent to minimizing $-I(X;Y)$, and finding the capacity can be cast as the following convex problem:

Minimize:

$$\sum_{j=1}^{n} (A^T p)_j \log (A^T p)_j - \sum_{i=1}^{n}\sum_{j=1}^{n} p_i A_{ij} \log A_{ij}.$$

Subject to:

$$\begin{cases} p \succeq \mathbf{0} \\ \mathbf{1}^T p = 1. \end{cases}$$

The optimal p^* can be found efficiently using various algorithms such as gradient methods [22], but in a few cases, p^* can be found directly using the Karush-Kuhn-Tucker (KKT) conditions [22]. To explain the KKT conditions, we first state the canonical convex optimization problem below:

Problem **P1**: Minimize: $f(x)$

Subject to:

$$\begin{cases} g_i(x) \leq 0, i = 1, 2, \dots n, \\ h_j(x) = 0, j = 1, 2, \dots, m, \end{cases}$$

where $f(x)$, $g_i(x)$ are convex functions and $h_j(x)$ is a linear function.

Define the Lagrangian function as:

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{n} \lambda_i g_i(x) + \sum_{j=1}^{m} \nu_j h_j(x), \tag{5}$$

then the KKT conditions [22] states that, the optimal point x^* must satisfy:

$$\begin{cases} g_i(x^*) \leq 0, \\ h_j(x^*) = 0, \\ \frac{dL(x,\lambda,\nu)}{dx}|_{x=x^*,\lambda=\lambda^*,\nu=\nu^*} = 0, \\ \lambda_i^* g_i(x^*) = 0, \\ \lambda_i^* \geq 0. \end{cases} \tag{6}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

B. Elementary Linear Algebra Results

We first begin with some definitions and preliminaries that will be used to derive our results.

Definition 1. *Let $A \in \mathbf{R}^{n\times n}$ be an invertible channel matrix and $H(A_i) = -\sum_{k=1}^{n} A_{ik} \log A_{ik}$ be the entropy of i^{th} row, define*

$$K_j = -\sum_{i=1}^{n} A_{ji}^{-1} \sum_{k=1}^{n} A_{ik} \log A_{ik} = \sum_{i=1}^{n} A_{ji}^{-1} H(A_i),$$

where A_{ji}^{-1} denotes the entry (j, i) of the inverse matrix A^{-1}. $K_{\max} = \max_j K_j$ and $K_{\min} = \min_j K_j$ are called the maximum and minimum inverse row entropies of A, respectively.

Definition 2. *Let $A \in \mathbf{R}^{n\times n}$ be a square matrix. The Gershgorin radius of i^{th} row of A [23] is defined as:*

$$R_i(A) = \sum_{j\neq i}^{n} |A_{ij}|. \tag{7}$$

The Gershgorin ratio of i^{th} row of A is defined as:

$$c_i(A) = \frac{A_{ii}}{R_i(A)}, \tag{8}$$

and the minimum Gershgorin ratio of A is defined as:

$$c_{\min}(A) = \min_i \frac{A_{ii}}{R_i(A)}. \tag{9}$$

We note that since the channel matrix is a stochastic matrix, therefore

$$c_{\min}(A) = \min_i \frac{A_{ii}}{R_i(A)} = \min_i \frac{A_{ii}}{1 - A_{ii}}. \tag{10}$$

Definition 3. *Let $A \in \mathbf{R}^{n\times n}$ be a square matrix.*
(a) A is called a positive matrix if $A_{ij} > 0\ \forall\ i, j$.
(b) A is called a strictly diagonally dominant positive matrix [24] if A is a positive matrix and

$$A_{ii} > \sum_{j\neq i} A_{ij}, \forall i, j. \tag{11}$$

Lemma 1. *Let $A \in \mathbf{R}^{n\times n}$ be a strictly diagonally dominant positive channel matrix then (a) it is invertible; (b) the eigenvalues of A^{-1} are $\frac{1}{\lambda_i}\ \forall\ i$ where λ_i are eigenvalues of A, (c) $A_{ii}^{-1} > 0$ and the largest absolute element in the i^{th} column of A^{-1} is A_{ii}^{-1}, i.e., $A_{ii}^{-1} \geq |A_{ji}^{-1}|\ \forall\ j$.*

Proof. The proof is shown in Appendix A. □

Lemma 2. *Let $A \in \mathbf{R}^{n\times n}$ be a strictly diagonally dominant positive matrix, then:*

$$c_i(A^{-T}) \geq \frac{c_{\min}(A) - 1}{(n-1)}, \forall i. \tag{12}$$

Moreover, for any rows k and l,

$$|A_{ki}^{-1}| + |A_{li}^{-1}| \leq A_{ii}^{-1} \frac{c_{\min}(A)}{c_{\min}(A) - 1}, \forall i. \tag{13}$$

Proof. The proof is shown in Appendix B. □

Lemma 3. *Let $A \in \mathbf{R}^{n\times n}$ be a strictly diagonally dominant positive matrix, then:*

$$\max_{i,j} A_{ij}^{-1} \leq \frac{1}{\sigma_{\min}(A)}, \tag{14}$$

where $\max_{i,j} A_{ij}^{-1}$ is the largest entry in A^{-1} and $\sigma_{\min}(A)$ is the minimum singular value of A.

Proof. The proof is shown in Appendix C. □

Lemma 4. *Let $A \in \mathbf{R}^{n\times n}$ be an invertible channel matrix, then*

$$A^{-1}\boldsymbol{1} = \boldsymbol{1},$$

i.e., the sum of any row of A^{-1} equals to 1. Furthermore, for any probability mass vector x, sum of the vector $y = A^{-T}x$ equal to 1.

Proof. The proof is shown in Appendix D. □

III. Main Results

Our first main result is an upper bound on the capacity of discrete memoryless channels having invertible positive channel matrices.

Proposition 1 (Main Result 1)**.** *Let $A \in \mathbf{R}^{n\times n}$ be an invertible positive channel matrix and*

$$q_j^* = \frac{2^{-K_j}}{\sum_{i=1}^{n} 2^{-K_i}}, \tag{15}$$

$$p' = A^{-T} q^*, \tag{16}$$

then the capacity C associated with the channel matrix A is upper bounded by:

$$C \leq -\sum_{j=1}^{n} q_j^* \log q_j^* + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i' A_{ij} \log A_{ij}. \tag{17}$$

Proof. Let q be the pmf of the output Y, then $q = A^T p$. Thus,

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \quad (18) \\ &= -\sum_{j=1}^{n} q_j \log q_j + \sum_{i}^{n} (A^{-T}q)_i \sum_{k}^{n} A_{ik} \log A_{ik}. \end{aligned}$$

We construct the Lagrangian in (5) using $-I(X;Y)$ as the objective function and optimization variable q_j:

$$L(q_j, \lambda_j, \nu_j) = -I(X;Y) - \sum_{j=1}^{n} q_j\lambda_j + \nu(\sum_{j=1}^{n} q_j - 1), \tag{19}$$

where the constraints $g(x)$ and $h(x)$ in problem **P1** are translated into $-q_j \leq 0$ and $\sum_{j=1}^{n} q_j = 1$, respectively.

Using the KKT conditions in (6), the optimal points q_j^*, λ_j^*, ν^* for all j, must satisfy:

$$q_j^* \geq 0, \tag{20}$$

$$\sum_{j=1}^{n} q_j^* = 1, \tag{21}$$

$$\nu^* - \lambda_j^* - \frac{dI(X;Y)}{dq_j^*} = 0, \tag{22}$$

$$\lambda_j^* \geq 0, \tag{23}$$

$$\lambda_j^* q_j^* = 0. \tag{24}$$

Since $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$, there exists at least one $p_i > 0$. Since $A_{ij} > 0 \; \forall i, j$, we have:

$$q_j^* = \sum_{i=1}^{n} p_i A_{ij} > 0, \forall j. \tag{25}$$

Based on (24) and (25), we must have $\lambda_j^* = 0, \forall j$. Therefore, all five KKT conditions (20-24) are reduced to the following two conditions:

$$\sum_{j=1}^{n} q_j^* = 1, \tag{26}$$

$$\nu^* - \frac{dI(X;Y)}{dq_j^*} = 0. \tag{27}$$

Next,

$$\begin{aligned} \frac{dI(X;Y)}{dq_j} &= \sum_{i=1}^{n} A_{ji}^{-1} \sum_{k=1}^{n} A_{ik} \log A_{ik} - (1 + \log q_j) \\ &= -K_j - (1 + \log q_j). \end{aligned} \tag{28}$$

Using (27) and (28), we have:

$$q_j^* = 2^{-K_j - \nu^* - 1}. \tag{29}$$

Plugging (29) to (26), we have:

$$\sum_{j=1}^{n} 2^{-K_j - \nu^* - 1} = 1,$$

$$\nu^* = \log \sum_{j=1}^{n} 2^{-K_j - 1}.$$

From (29),

$$q_j^* = 2^{-K_j - \nu^* - 1} = \frac{2^{-K_j}}{2^{\nu^* + 1}} = \frac{2^{-K_j}}{\sum_{j=1}^{n} 2^{-K_j}}, \forall j. \tag{30}$$

We know that a valid optimal input distribution has to satisfy $0 \leq p_i^* \leq 1$ and $\sum_{i=1}^{n} p_i^* = 1$. If q^* is such that $p' = A^{-T} q^* \succeq 0$ and $(A^{-T} q^*)^T \mathbf{1} = \sum_{i=1}^{n} p_i' = 1$, then $p' = p^*$ is a valid and optimal pmf, and Proposition 1 will hold with equality by the KKT conditions. Now, the condition $\sum_{i=1}^{n} p_i' = 1$ holds by Lemma 4. However, the condition $0 \leq p_i' \leq 1$ may not satisfy. In this case, maximizing $I(X;Y)$ in terms of q and ignoring this constraint is equivalent to enlarging the feasible region. Since $\max_{x \in A} f(x) \geq \max_{x \in B} f(x)$ if $B \subset A$ for any arbitrary $f(x)$, the upper bound of channel capacity in Proposition 1 is achieved by plugging q^* from (15) into (16) to obtain p', and plugging p' and q^* into (4). $\square$

We note that the closed-form expressions for channel capacity are also described in [4] and [6] (Section 3.3). However in both [4] and [6], the sufficient conditions for the closed-form expressions are not fully characterized. We now show another contribution that characterizes the sufficient conditions on the channel matrix A such that its capacity can be written in closed-form expression, specifically the upper bound in (17).

Proposition 2 (Main Result 2). *Let $A \in \mathbf{R}^{n \times n}$ be a strictly diagonally dominant positive matrix, if $\forall i$,*

$$c_i(A^{-T}) \geq (n-1) 2^{K_{\max} - K_{\min}}, \tag{31}$$

then the capacity of the channel having channel matrix A admits a closed-form expression which is exactly the upper bound in Proposition 1.

Proof. Based on the discussion of the KKT conditions, it is sufficient to show that if $p^* = A^{-T} q^* \succeq 0$ and $\sum_{i=1}^{n} p_i^* = (A^{-T} q^*)^T \mathbf{1} = 1$ then C has a closed-form expression. The condition $(A^{-T} q^*)^T \mathbf{1} = 1$ is always true as shown in Lemma 4 in the Appendix D. Thus, we only need to show that if $c_i(A^{-T}) \geq 2^{K_{\max} - K_{\min}}$, then $p^* = A^{-T} q^* \succeq 0$.

Let $q_{\min}^* = \min_j q_j^*$ and $q_{\max}^* = \max_j q_j^*$, we have:

$$\begin{aligned} p_i^* &= \sum_j q_j^* A_{ji}^{-1} \\ &= q_i^* A_{ii}^{-1} + \sum_{j \neq i} q_j^* A_{ji}^{-1} \\ &\geq q_{\min}^* A_{ii}^{-1} - (\sum_{j \neq i} q_j^*)(\sum_{j \neq i} |A_{ji}^{-1}|) \quad (32) \\ &\geq q_{\min}^* A_{ii}^{-1} - (n-1) q_{\max}^* (\sum_{j \neq i} |A_{ji}^{-1}|), \quad (33) \end{aligned}$$

with (32) due to $A_{ii}^{-1} > 0$ which follows by Lemma 1-(c), (33) is due to $q_{\max}^* \geq q_j^* \; \forall \; j$. Now if we want $p_i^* \geq 0, \; \forall \; i$, from (33), it is sufficient to require that, $\forall i$,

$$\begin{aligned} c_i(A^{-T}) = \frac{A_{ii}^{-1}}{\sum_{j \neq i} |A_{ji}^{-1}|} &\geq \frac{(n-1) q_{\max}^*}{q_{\min}^*} \\ &= (n-1) \frac{\frac{2^{-K_{\min}}}{\sum_{j=1}^{n} 2^{-K_j}}}{\frac{2^{-K_{\max}}}{\sum_{j=1}^{n} 2^{-K_j}}} \quad (34) \\ &= (n-1) 2^{K_{\max} - K_{\min}}, \end{aligned}$$

with (34) due to (30) and $q_{\max}^*$, $q_{\min}^*$ are corresponding to $K_{\min}$, $K_{\max}$, respectively. Thus, Proposition 2 is proven. $\square$

We are now ready to state and prove the third main result that characterizes the sufficient conditions on a channel matrix so that the upper bound in Proposition 1 is precisely the capacity.

Proposition 3. *Let $A \in \mathbf{R}^{n\times n}$ be a strictly diagonally dominant positive channel matrix and $H_{\max}(A)$ be the maximum row entropy of A. The capacity C is the upper bound in Proposition 1 i.e., hold with equality if*

$$\sqrt[V]{\frac{c_{\min}(A)-1}{(n-1)^2}} \geq 2^{\frac{nH_{\max}(A)}{\sigma_{\min}(A)}}, \tag{35}$$

where $\sigma_{\min}(A)$ is the minimum singular value of channel matrix A, and

$$V = \frac{c_{\min}(A)}{c_{\min}(A)-1}. \tag{36}$$

Proof. From (12) in Lemma 2 and Proposition 2, if we can show that

$$\frac{c_{\min}(A)-1}{(n-1)} \geq (n-1)2^{K_{\max}-K_{\min}}, \tag{37}$$

then Proposition 3 is proven. Suppose that $K_{\max}$ and $K_{\min}$ are obtained at rows $j=L$ and $j=S$, respectively. We note that from (30), $q_{\max} = \max_j q_j$ and $q_{\min} = \min_j q_j$ correspond to $K_{\min}$ and $K_{\max}$, respectively. Thus, from the Definition **1**, we have:

$$\begin{aligned}
K_{\max}-K_{\min} &= \sum_{i=1}^{n} A^{-1}_{Li}H(A_i) - \sum_{i=1}^{n} A^{-1}_{Si}H(A_i) \\
&\leq |\sum_{i=1}^{n} A^{-1}_{Li}H(A_i)| + |\sum_{i=1}^{n} A^{-1}_{Si}H(A_i)| \quad (38)\\
&\leq \sum_{i=1}^{n} |A^{-1}_{Li}||H(A_i)| + \sum_{i=1}^{n} |A^{-1}_{Si}||H(A_i)| \quad (39)\\
&\leq H_{\max}(A)\sum_{i=1}^{n}(|A^{-1}_{Li}| + |A^{-1}_{Si}|) \quad (40)\\
&\leq H_{\max}(A)\sum_{i=1}^{n} A^{-1}_{ii}\frac{c_{\min}(A)}{c_{\min}(A)-1} \quad (41)\\
&\leq nH_{\max}(A)(\max_{i,j} A^{-1}_{ij})\frac{c_{\min}(A)}{c_{\min}(A)-1} \quad (42)\\
&\leq \frac{nH_{\max}(A)V}{\sigma_{\min}(A)}, \quad (43)
\end{aligned}$$

where (38) due to the property of absolute value function, (39) due to Schwarz inequality, (40) due to $H_{\max}(A)$ is the maximum row entropy of A, (41) due to (13), (42) due to $\max_{i,j} A^{-1}_{ij}$ is the largest entry in A^{-1} and (43) is due to Lemma 3. Thus,

$$(n-1)2^{\frac{nH_{\max}(A)V}{\sigma_{\min}(A)}} \geq (n-1)2^{K_{\max}-K_{\min}}. \tag{44}$$

From (37) and (44), if

$$\frac{c_{\min}(A)-1}{(n-1)} \geq (n-1)2^{\frac{nH_{\max}(A)V}{\sigma_{\min}(A)}}, \tag{45}$$

then the capacity C is the upper bound in Proposition 1. (45) is equivalent to (35). Thus Proposition 3 is proven. ☐

We note that the condition in Proposition 3 is easier to verify than the condition in Proposition 2 since it can be performed without requiring matrix inverse. Other easy-to-use versions of checking condition are stated in Proposition 4 and Corollary 1.

Proposition 4. *The capacity C is the upper bound in Proposition 1 if*

$$\frac{c_{\min}(A)-1}{(n-1)^2} \geq 2^{\frac{2n\log n}{\sigma_{\min}(A)}}. \tag{46}$$

Proof. Similar to Proposition 3,

$$\begin{aligned}
K_{\max}-K_{\min} &\leq H_{\max}(A)\sum_{i=1}^{n}(|A^{-1}_{Li}| + |A^{-1}_{Si}|) \quad (47)\\
&\leq H_{\max}(A)n(2\max_{i,j} A^{-1}_{ij}) \quad (48)\\
&\leq \frac{2n\log n}{\sigma_{\min}(A)}, \quad (49)
\end{aligned}$$

with (47) is identical to (40), (48) is due to $\max_{i,j} A^{-1}_{ij}$ is the largest entry in A^{-1}, (49) due to $H_{\max}(A) \leq \log n$ and Lemma 3. Thus, by changing $\frac{nH_{\max}(A)V}{\sigma_{\min}(A)}$ in (45) by $\frac{2n\log n}{\sigma_{\min}(A)}$, the Proposition 4 is proven. ☐

A direct result of Proposition 3 without using singular value is shown in Corollary 1.

Corollary 1. *The capacity C is the upper bound in Proposition 1 if*

$$\sqrt[V]{\frac{c_{\min}(A)-1}{(n-1)^2}} \geq 2^{\frac{nH^*_{\max}(A)}{\sigma^*}}, \tag{50}$$

where,

$$V = \frac{c_{\min}(A)}{c_{\min}(A)-1}, \tag{51}$$

$$\sigma^* = \frac{c_{\min}(A)-n/2}{c_{\min}(A)+1}, \tag{52}$$

$$H^*_{\max}(A) = \log(c_{\min}(A)+1) + \frac{\log(n-1) - c_{\min}(A)\log c_{\min}(A)}{c_{\min}(A)+1}. \tag{53}$$

Proof. We will construct the lower bound for $\sigma_{\min}(A)$ and the upper bound for $H_{\max}(A)$. From Lemma 5 in Appendix E

$$\sigma_{\min}(A) \geq \frac{c_{\min}(A)-n/2}{c_{\min}(A)+1} = \sigma^*, \tag{54}$$

and

$$\begin{aligned}
H_{\max}(A) &\leq \log(c_{\min}(A)+1) + \frac{\log(n-1) - c_{\min}(A)\log c_{\min}(A)}{c_{\min}(A)+1}\\
&= H^*_{\max}(A). \quad (55)
\end{aligned}$$

Therefore

$$\frac{nH_{\max}(A)V}{\sigma_{\min}(A)} \leq \frac{nH^*_{\max}(A)}{\sigma^*}. \tag{56}$$

Thus, by changing $\frac{nH_{\max}(A)}{\sigma_{\min}(A)}$ in (35) by $\frac{nH^*_{\max}(A)}{\sigma^*}$, the Corollary 1 is proven.

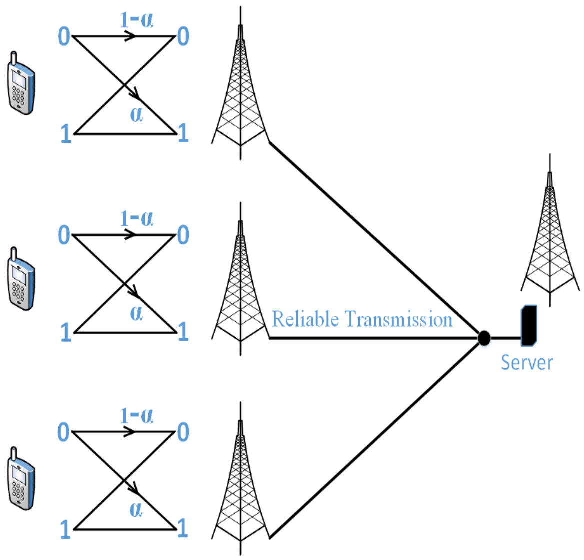

Figure 1. Relay-MISO channel

We note that, when $c_{\min}(A)$ is relatively larger than the size of matrix n, the lower bound of $\sigma_{\min}(A)$ goes to 1. We also note that (50) can be checked efficiently without requiring both $H_{\max}(A)$ and $\sigma_{\min}(A)$ at the expense of a looser upper bound as compare to (35). $\square$

IV. Examples and Numerical Results

A. Example 1: Cooperative Relay-MISO Channels

In this example, we investigate the channel capacity for a class of channels named Relay-MISO (Relay - Multiple Input Single Output). Relay-MISO channel [25] can be constructed by the combination of a relay channel [26] [27] and a Multiple Input Single Output channel, as illustrated in Fig. 1.

In a Relay-MISO channel, n senders want to transmit data to a same receiver via n relay base station nodes. The uplink of these senders using wireless links that are prone to transmission errors. Each sender can transmit bit "0" or "1" with the probability of bit flipping is α, $0 \leq \alpha \leq 1$. For a simplicity, suppose that n relay channels have the same error probability α. Next, all of the relay base station nodes will relay the signal by a reliable channel such as optical fiber cable to a same receiver. The receiver adds all the relay signals (symbols) to produce a single output symbol.

It can be shown that the channel matrix of this Relay-MISO channel [25] is an invertible matrix of size $(n+1)\times(n+1)$ whose A_{ij} can be computed as:

$$A_{ij} = \sum_{s=\max(i-j,0)}^{s=\min(n+1-j,i-1)} \binom{j-i+s}{n+1-i}\binom{s}{i-1}\alpha^{j-i+2s}(1-\alpha)^{n-(j-i+2s)}.$$

We note that this Relay-MISO channel matrix is invertible and the inverse matrix has the closed-form expression which is characterized in [25]. For example, the channel matrix of a Relay-MISO channel with $n=3$ is given as follows:

$$\begin{bmatrix} (1-\alpha)^3 & 3(1-\alpha)^2\alpha & 3(1-\alpha)\alpha^2 & \alpha^3 \\ \alpha(1-\alpha)^2 & 2\alpha^2(1-\alpha)+(1-\alpha)^3 & 2(1-\alpha)^2\alpha+\alpha^3 & (1-\alpha)\alpha^2 \\ (1-\alpha)\alpha^2 & 2(1-\alpha)^2\alpha+\alpha^3 & 2\alpha^2(1-\alpha)+(1-\alpha)^3 & \alpha(1-\alpha)^2 \\ \alpha^3 & 3(1-\alpha)\alpha^2 & 3(1-\alpha)^2\alpha & (1-\alpha)^3 \end{bmatrix},$$

where $0 \leq \alpha \leq 1$. We note that this channel matrix is strictly diagonally dominant matrix when α is close to 0 or α is close to 1. In addition, for α values that are close to 0 or 1, it can be shown that channel matrix A satisfies the conditions in Proposition 3. Thus, the channel capacity admits a closed-form expression in Proposition 1. For other values of α, e.g. closer to 0.5, the optimality conditions in Proposition 3 no longer holds. In this case, Proposition 1 can still be used as a good upper bound on the capacity.

We show that our upper bound is tighter than existing upper bounds. In particular, Fig. 2 shows the actual capacity and the known upper bounds as functions of parameter α for Relay-MISO channels having $n=3$. The green curve depicts the actual capacity computed using convex optimization algorithm. The red curve is constructed using our closed-form expression in Proposition 1, and the blue dotted curve is the constructed using the well-known upper bound result of channel capacity in [11], [28]. Specifically, this upper bound is:

$$C \leq \log(\sum_{j=1}^{n} \max_i A_{ij}). \tag{57}$$

Finally, the red dotted curve shows another well-known upper bound by Arimoto [3] which is:

$$C \leq \log(n) + \max_j [\sum_{i=1}^{n} A_{ji} \log(\frac{A_{ji}}{\sum_{k=1}^{n} A_{ki}})]. \tag{58}$$

We note that the second term is negative.

Fig. 2 shows that our closed-form upper bound is precisely the capacity (the red and green graphs are overlapped) when α values are close to 0 or 1 as predicted by the optimality conditions in Proposition 3. On the other hand, when α values are closer to 0.5, our optimality conditions no longer hold. In this case, we can only determine the upper bound. However, it is interesting to note that our upper bound in this case is tighter than both the Boy-Chiang [11] and Arimoto [3] upper bounds.

B. Example 2: Symmetric and Weakly Symmetric Channels

Our results confirm the capacity of the well known symmetric and weakly symmetric channel matrices. In particular, when the channel matrix is symmetric and positive definite, all our results are applicable. Indeed, since the channel matrix is symmetric and positive definite, the inverse channel matrix exists and also is symmetric. From Definition 1, all values of K_j is the same since they are the same sum of permutation entries. Therefore, from Proposition 1, the optimal output probability mass vector

$$q_j^* = \frac{2^{-K_j}}{\sum_{i=1}^{n} 2^{-K_i}} \tag{59}$$

This is the author's version of an article that has been published in this journal. Changes were made to this version by the publisher prior to publication.
The final version of record is available at http://dx.doi.org/10.1109/TVT.2020.3003560

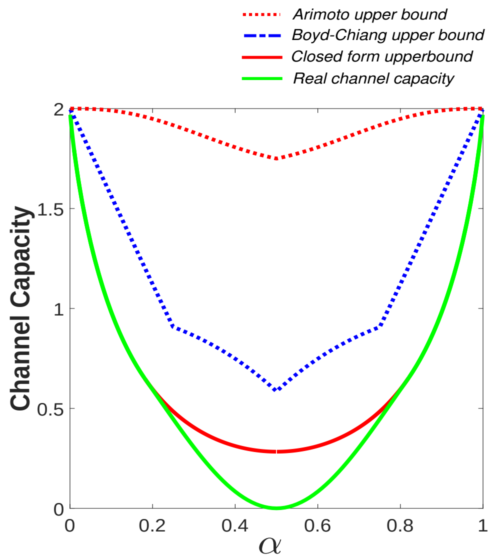

Figure 2. Channel capacity and various upper bounds as functions of α

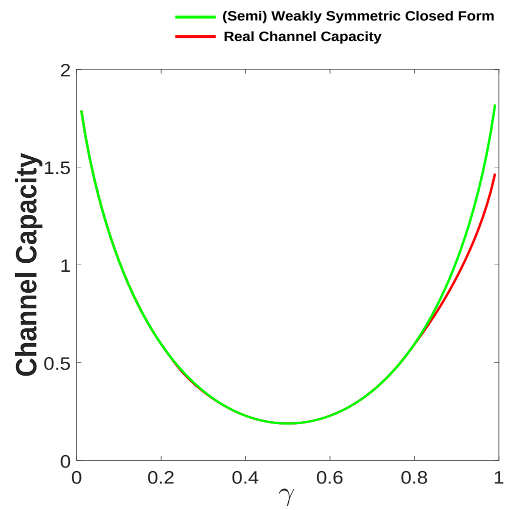

Figure 3. Channel capacity of (semi) weakly symmetric channel as a function of γ

are equal each other for all j. As a result, the input probability mass function $p^* = A^{-T}q^*$ is the uniform distribution, and the channel capacity is upper bounded by:

$$C \leq -\sum_{j=1}^{n} q_j^* \log q_j^* + \sum_{i=1}^{n}\sum_{j=1}^{n} p_i^* A_{ij} \log A_{ij} \quad (60)$$

$$= \log n - H(A_{row}). \quad (61)$$

Interestingly, our result also shows the capacities of many channels that are *not* weakly symmetric, but admits the closed-form formula of weakly symmetric channels. In particular, consider a channel matrix called semi-weakly symmetric whose all rows are permutations of each other, but the sum of entries in each column might not be the same. Furthermore, if the optimal condition is satisfied (Proposition 3), then the channel has closed-form capacity which is identical to the capacity of a symmetric and weakly symmetric channel:

$$C = \log n - H(A_{row}). \quad (62)$$

Note that every row of a quasi-symmetric matrix is a permutation of the first row [29]. Thus, a quasi-symmetric matrix is an example of a semi-weakly symmetric matrix. For example, the following channel matrix:

$$A = \begin{bmatrix} 0.93 & 0.04 & 0.03 \\ 0.04 & 0.93 & 0.03 \\ 0.04 & 0.03 & 0.93 \end{bmatrix}$$

is not a weakly symmetric channel even though its rows are permutations of each other since the column sums are different. However, this channel matrix satisfies Proposition 3 and Corollary 1 since $n = 3$, $\sigma_{\min}(A) = 0.88916$, $\sigma^* = 0.825$, $H_{\max}(A) = 0.43489$, $H^*_{\max}(A) = 0.43592$ and $c_{\min}(A) = 13.286$. Thus, it has closed-form formula for capacity, and can be easily shown to be $C = \log 3 - H(0.93, 0.04, 0.03) = 1.1501$. The optimal output and input probability mass vectors can be shown to be:

$$q^T = \begin{bmatrix} 0.33333 & 0.33333 & 0.33333 \end{bmatrix},$$

$$p^T = \begin{bmatrix} 0.32959 & 0.33337 & 0.33704 \end{bmatrix},$$

respectively.

The following channel matrix is another example of semi-weakly symmetric matrix whose entries are controlled by a parameter γ in the range of $(0, 1)$ and given by the following form:

$$\begin{bmatrix} (1-\gamma)^3 & 3(1-\gamma)^2\gamma & 3(1-\gamma)\gamma^2 & \gamma^3 \\ 3(1-\gamma)^2\gamma & (1-\gamma)^3 & \gamma^3 & 3(1-\gamma)\gamma^2 \\ \gamma^3 & 3(1-\gamma)\gamma^2 & (1-\gamma)^3 & 3(1-\gamma)^2\gamma \\ \gamma^3 & 3(1-\gamma)\gamma^2 & 3(1-\gamma)^2\gamma & (1-\gamma)^3 \end{bmatrix}.$$

Fig. 3 shows the capacity upper bound of the semi-weakly symmetric channel and the actual channel capacity as function of γ. Theoretically, the conditions in Proposition 3 and Proposition 4 can be shown to hold for $\gamma \leq 0.02$. However, for much values of γ, the upper bound is identical to the actual channel capacity which can be numerically determined using CVX [12]. This happens because these conditions are sufficient but not necessary.

C. Example 3: Unreliable Channels

We now consider an unreliable channel whose channel matrix is:

$$A = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.7 & 0.1 & 0.2 \\ 0.5 & 0.05 & 0.45 \end{bmatrix}.$$

In this case, our optimality conditions do not satisfy, and the Arimoto upper bound is tightest (0.17083) as compared to our upper bound (0.19282) and Boyd-Chiang upper bound (0.848).

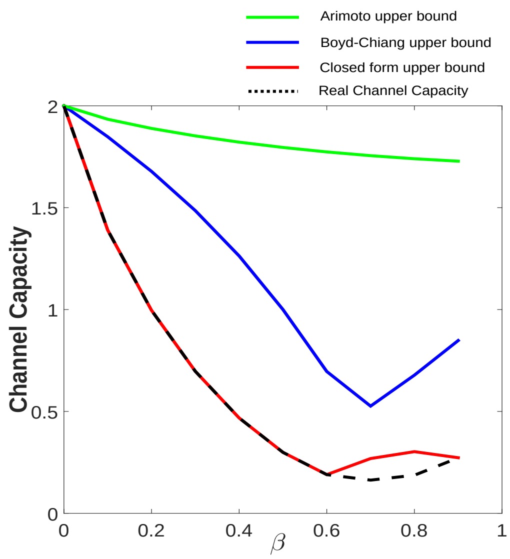

Figure 4. Channel capacity and various upper bounds functions of β

D. Example 4: Bounds as Function of Channel Reliability

Since we know that our proposed bounds are tight if the channel is reliable, we want to examine quantitatively how channel reliability affects various bounds. In this example, we consider a special class of channel whose channel matrix entries are controlled by a reliability parameter β for $0 \leq \beta \leq 1$ as shown below:

$$A = \begin{bmatrix} 1-\beta & 0.3\beta & 0.4\beta & 0.3\beta \\ 0.4\beta & 1-\beta & 0.3\beta & 0.3\beta \\ 0.5\beta & 0.4\beta & 1-\beta & 0.1\beta \\ 0.1\beta & 0.2\beta & 0.7\beta & 1-\beta \end{bmatrix}.$$

When β is small, the channel tends to be reliable and when β is large, the channel tends to be unreliable. Fig. 4 shows various upper bounds as a function of β together with the actual capacity. The actual channel capacities for various β are numerically computed using a convex optimization algorithm [12]. As seen, our closed-form upper bound expression for capacity (red curve) from Proposition 1 is much closer to the actual capacity (black dash curve) than other bounds for most values of β. When β is small ($\beta \leq 0.6$) or channel is reliable, the closed-form upper bound is precise the real channel capacity, and we can verify that the optimal conditions in Proposition 3 holds. When the channel becomes unreliable, i.e., $\beta \geq 0.6$, our upper bound is no longer tight, however, it is still the tightest among all the existing upper bounds. We note that when the β is small, the channel matrix becomes a nearly diagonally dominant matrix, and our upper bound is tightest.

V. Conclusion

In this paper, we describe an elementary technique based on Karush-Kuhn-Tucker (KKT) conditions to obtain (1) a good upper bound of a discrete memoryless channel having an invertible positive channel matrix and (2) a closed-form expression for the capacity if the channel matrix satisfies certain conditions related to its singular value and its Gershgorin's disk. We provide a number of channels where the proposed upper bound becomes precisely the capacity. We also demonstrate that our proposed bounds are tighter than other existing bounds for these channels.

Appendix

A. Proof of Lemma 1

For claim (a), since the channel matrix is strictly diagonally dominant, using Gershgorin circle theorem [23] that for any eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, we must have:

$$\lambda_i \geq A_{ii} - \sum_{j \neq i} |A_{ij}| > 0.$$

Thus, $det(A) = \lambda_1 \lambda_2 \ldots \lambda_n > 0$. Therefore, A is invertible.

Claim (b) is a well-known algebra result [30].

For claim (c), due to $AA^{-1} = I$ and $A_{ij} > 0 \; \forall \; i, j$, therefore, $\forall \; j$ exists at least i such that $A^{-1}_{ij} \neq 0$. Therefore the largest absolute entry in each column $\neq 0$. Claim (c) can be obtained by contradiction. Suppose that the largest absolute entry in j^{th} column of A^{-1} is A^{-1}_{ij} in i^{th} row, that said $|A^{-1}_{ij}| \geq |A^{-1}_{kj}| \; \forall \; k$. We suppose that $A^{-1}_{ij} < 0$. Thus:

$$\begin{aligned} \sum_{k=1}^{n} A_{ik} A^{-1}_{kj} &\leq -A_{ii}|A^{-1}_{ij}| + \sum_{k=1, k\neq i}^{n} A_{ik}|A^{-1}_{ij}| \quad (63) \\ &= (-A_{ii} + \sum_{k=1, k\neq i}^{n} A_{ik})|A^{-1}_{ij}| \\ &< 0, \quad (64) \end{aligned}$$

which contradicts with $\sum_{k=1}^{n} A_{ik} A^{-1}_{kj} = I_{ij} \geq 0$. Thus, the largest absolute value in each column of A^{-1} is positive. That said in j^{th} column, if $|A^{-1}_{ij}| \geq |A^{-1}_{kj}| \; \forall \; k$, then $A^{-1}_{ij} > 0$.

Now, suppose that the largest absolute element in j^{th} column of A^{-1}, is A^{-1}_{ij} with $i \neq j$ and $A^{-1}_{ij} > 0$. Then:

$$\begin{aligned} 0 &= \sum_{k=1}^{n} A_{ik} A^{-1}_{kj} \\ &\geq A_{ii}|A^{-1}_{ij}| - \sum_{k=1, k\neq i}^{n} A_{ik}|A^{-1}_{ij}| \quad (65) \\ &= (A_{ii} - \sum_{k=1, k\neq i}^{n} A_{ik}) A^{-1}_{ij} \\ &> 0, \quad (66) \end{aligned}$$

with (65) due to A^{-1}_{ij} is the largest absolute element in j^{th} column and (66) due to A is strictly diagonally dominant matrix. This is a contradiction. Therefore, the largest absolute entry in j^{th} column of A^{-1} should be A^{-1}_{jj} and $A^{-1}_{jj} > 0$.

B. Proof of Lemma 2

First, let's show that the second largest absolute value in each column of A^{-1} is a negative entry by contradiction method. Suppose that the second largest absolute value in j^{th} column of A^{-1} is positive and in k^{th} row ($k \neq j$), $A^{-1}_{kj} \geq 0$. Consider,

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} A_{ki}A^{-1}_{ij} \\
&\geq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} - |\sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}A^{-1}_{ij}| \quad (67)\\
&\geq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} - \sum_{i=1,i\neq k;i\neq j}^{n} |A_{ki}A^{-1}_{ij}| \quad (68)\\
&\geq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} - \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}|A^{-1}_{ij}| \quad (69)\\
&\geq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} - \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}|A^{-1}_{kj}| \quad (70)\\
&= A_{kj}A^{-1}_{jj} + A^{-1}_{kj}(A_{kk} - \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}) \quad (71)\\
&> 0, \quad (72)
\end{aligned}
$$

with (67) due to the fact that $C \geq -|C| \; \forall \; C$, (68) due to the triangle inequality, (69) due to A_{ki} is positive, (70) due to A^{-1}_{kj} is the second largest absolute value in j^{th} column of A^{-1}, (71) due to the assumption that $A^{-1}_{kj} \geq 0$ and (72) due to (11) such that $A_{kk} \geq \sum_{i=1,i\neq k}^{n} A_{ki} \geq \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}$. Thus, the second largest absolute value in column of A^{-1} is negative ($A^{-1}_{kj} < 0$). Due to Lemma 1 part (c), A^{-1}_{jj} is the largest absolute value entry and $A^{-1}_{jj} > 0$. Similarly,

$$
\begin{aligned}
0 &= \sum_{i=1}^{n} A_{ki}A^{-1}_{ij} \\
&\leq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} + |\sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}A^{-1}_{ij}| \quad (73)\\
&\leq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} + \sum_{i=1,i\neq k;i\neq j}^{n} |A_{ki}A^{-1}_{ij}| \quad (74)\\
&\leq A_{kj}A^{-1}_{jj} + A_{kk}A^{-1}_{kj} + \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}|A^{-1}_{ij}| \quad (75)\\
&\leq A_{kj}A^{-1}_{jj} - A_{kk}|A^{-1}_{kj}| + \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}|A^{-1}_{kj}|, \quad (76)
\end{aligned}
$$

with (73) due to the fact that $C \leq |C| \; \forall \; C$, (74) due to the triangle inequality, (75) due to $A_{ki} \geq 0, \; \forall \; i$ and (76) due to $A^{-1}_{kj} < 0$ and A^{-1}_{kj} is the second largest absolute value in j^{th} column. Hence,

$$
\begin{aligned}
A_{kj}A^{-1}_{jj} &\geq A_{kk}|A^{-1}_{kj}| - \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki}|A^{-1}_{kj}| \\
A^{-1}_{jj} &\geq \frac{|A^{-1}_{kj}|(A_{kk} - \sum_{i=1,i\neq k;i\neq j}^{n} A_{ki})}{A_{kj}} \\
A^{-1}_{jj} &\geq |A^{-1}_{kj}| \frac{A_{kk} - \dfrac{A_{kk}}{c_{\min}(A)}}{\dfrac{A_{kk}}{c_{\min}(A)}} \quad (77)\\
A^{-1}_{jj} &\geq |A^{-1}_{kj}|[c_{\min}(A) - 1], \forall j, \quad (78)
\end{aligned}
$$

with (77) due to Definition 2 and (9) such that $\frac{A_{kk}}{c_{\min}(A)} \geq \sum_{i=1,i\neq k}^{n} A_{ki} \geq \sum_{i=1,i\neq k,i\neq j}^{n} A_{ki}$. Thus, we have:

$$
c_j(A^{-T}) = \frac{A^{-1}_{jj}}{\sum_{k\neq j} |A^{-1}_{kj}|} \geq \frac{c_{\min}(A) - 1}{n-1}. \quad (79)
$$

Thus, (12) is proven.

Next, we note that from (78)

$$
\frac{A^{-1}_{jj}}{c_{\min}(A) - 1} \geq |A^{-1}_{kj}|, \forall k. \quad (80)
$$

Moreover, from Lemma 1, $A^{-1}_{jj} \geq 0$ and is the largest entry in j^{th} row. Thus, for an arbitrary L and S,

$$
\begin{aligned}
|A^{-1}_{Lj}| + |A^{-1}_{Sj}| &\leq A^{-1}_{jj} + \frac{A^{-1}_{jj}}{c_{\min}(A) - 1} \\
&= A^{-1}_{jj} \frac{c_{\min}(A)}{c_{\min}(A) - 1}, \forall j. \quad (81)
\end{aligned}
$$

Thus, (13) is proven.

C. Proof of Lemma 3

Consider the matrix $B = A^{-1}A^{-T}$, B is symmetric, all its eigenvalues are real and satisfy the Rayleigh quotient [31]. Let λ_B^{max} be the maximum eigenvalue of B then from [31]

$$
R(B, x) = \frac{x^* B x}{x^* x} \leq \lambda_B^{max}. \quad (82)
$$

Consider the unit vector $e = [0, \ldots, 1, \ldots, 0]^T$ with entry "1" is in the i^{th} column. Let $x = e$ in (82), we have:

$$
B_{ii} \leq \lambda_B^{max}. \quad (83)
$$

Thus,

$$
\begin{aligned}
\lambda_B^{max} &\geq B_{ii} \\
&= \sum_{j=1}^{n} A^{-1}_{ij}A^{-1}_{ij} \\
&\geq (A^{-1}_{ii})^2. \quad (84)
\end{aligned}
$$

Now since B is a symmetric matrix $\lambda_B^{max} = \sigma_{\max}(B)$ [30]. However, from [30], $\sigma_{\max}(B) = \sigma_{\max}(A^{-1}A^{-T}) = \sigma^2_{\max}A^{-1}$ and $\sigma_{\max}A^{-1} = \frac{1}{\sigma_{\min}(A)}$. Thus:

$$
\frac{1}{\sigma_{\min}(A)} \geq A^{-1}_{ii}. \quad (85)
$$

From Lemma 1-(c), the largest entry in A^{-1} must be a diagonal element, thus

$$\max_{i,j} A_{ij}^{-1} \leq \frac{1}{\sigma_{\min}(A)}.$$

D. Proof of Lemma 4

For the first claim, since A is a stochastic matrix,

$$A\mathbf{1} = \mathbf{1}.$$

Left multiply both sides by A^{-1} results in $\mathbf{1} = A^{-1}\mathbf{1}$. For the second claim, left multiplying $y = A^{-T}x$ by $\mathbf{1}^T$, we have:

$$\mathbf{1}^T y = \mathbf{1}^T A^{-T} x = x^T A^{-1}\mathbf{1} = x^T\mathbf{1} = 1,$$

where we use $A^{-1}\mathbf{1} = \mathbf{1}$ in the previous claim.

Thus, we have $\sum_{i=1}^{n} p_i^* = 1$ since from (30), q^* is a probability mass vector.

E. Proof of Corollary 1

Lemma 5. *Lower bound of $\sigma_{\min}(A)$ and upper bound of $H_{\max}(A)$ are σ^* and $H_{\max}^*(A)$, respectively*

$$\sigma_{\min}(A) \geq \sigma^* = \frac{c_{\min}(A) - n/2}{c_{\min}(A) + 1}, \tag{86}$$

and

$$H_{\max}(A) \leq H_{\max}^*(A), \tag{87}$$

where

$$H_{\max}^*(A) = \log(c_{\min}(A)+1) + \frac{\log(n-1) - c_{\min}(A)\log c_{\min}(A)}{c_{\min}(A)+1}. \tag{88}$$

Proof. Due to the channel matrix is a strictly diagonally dominant positive matrix. Thus, we have

$$A_{kk} \geq \frac{c_{\min}(A)}{c_{\min}(A)+1}, \tag{89}$$

$$R_k(A) = 1 - A_{kk} \leq 1 - \frac{c_{\min}(A)}{c_{\min}(A)+1} = \frac{1}{c_{\min}(A)+1}, \tag{90}$$

$$C_k(A) = \sum_{j=1, j\neq k}^{j=n} A_{jk} \leq \sum_{j=1, j\neq k}^{j=n} R_j(A) \leq \frac{n-1}{c_{\min}(A)+1}, \forall k, \tag{91}$$

with (89) due to (10), (90) due to (89), (91) due to the fact that $\forall\ j \neq k$, $A_{jk} \leq \sum_{j\neq k} A_{jk} = R_j(A)$ and each $R_j(A) \leq \frac{1}{c_{\min}(A)+1}$ which is proven in (89). Now, we are ready to establish the upper bound of $H_{\max}(A)$ and the lower bound of $\sigma_{\min}(A)$, respectively.

- Suppose that $H_{\max}(A)$ achieves at k^{th} row, then

$$\begin{aligned}
H_{\max}(A) &= -(\sum_{i=1}^{n} A_{ki}\log A_{ki}) \\
&= -(A_{kk}\log A_{kk} + \sum_{i=1, i\neq k}^{n} A_{ki}\log A_{ki}) \\
&= -A_{kk}\log A_{kk} \\
&- (1-A_{kk}) \sum_{i=1, i\neq k}^{n} \frac{A_{ki}}{1-A_{kk}} (\log \frac{A_{ki}}{1-A_{kk}} + \log(1-A_{kk})) \\
&= -A_{kk}\log A_{kk} \\
&- (1-A_{kk}) \sum_{i=1, i\neq k}^{n} \frac{A_{ki}}{1-A_{kk}} \log \frac{A_{ki}}{1-A_{kk}} \\
&- (1-A_{kk})\log(1-A_{kk}) \\
&\leq -A_{kk}\log A_{kk} + (1-A_{kk})\log(n-1) \\
&- (1-A_{kk})\log(1-A_{kk}) \qquad (92) \\
&= -(A_{kk}\log A_{kk} + (1-A_{kk})\log(\frac{1-A_{kk}}{n-1})) \\
&\leq -(\frac{c_{\min}(A)}{c_{\min}(A)+1}\log\frac{c_{\min}(A)}{c_{\min}(A)+1} \\
&+ (1-\frac{c_{\min}(A)}{c_{\min}(A)+1})\log\frac{1-\frac{c_{\min}(A)}{c_{\min}(A)+1}}{n-1}) \qquad (93) \\
&= \log(c_{\min}(A)+1) + \frac{\log(n-1) - c_{\min}(A)\log c_{\min}(A)}{c_{\min}(A)+1},
\end{aligned}$$

with (92) is due to $-\sum_{i=1, i\neq k}^{n} \frac{A_{ki}}{1-A_{kk}} \log \frac{A_{ki}}{1-A_{kk}}$ is the entropy of $n-1$ elements which is bounded by $\log(n-1)$. For (93), first we show that $f(x) = -(x\log x + (1-x)\log(\frac{1-x}{n-1}))$ is monotonically decreasing function for $\frac{x}{1-x} \geq n-1$. Indeed,

$$\begin{aligned}
\frac{d(f(x))}{d(x)} &= \log x - \log(1-x) - \log(n-1) \\
&= -(\log\frac{x}{1-x} - \log(n-1)).
\end{aligned}$$

Thus, if $\frac{x}{1-x} \geq n-1$ then $\frac{d(f(x))}{d(x)} \leq 0$. However, from (89),

$$\frac{A_{kk}}{1-A_{kk}} \geq \frac{\frac{c_{\min}(A)}{c_{\min}(A)+1}}{1-\frac{c_{\min}(A)}{c_{\min}(A)+1}} = c_{\min}(A). \tag{94}$$

From (50)

$$c_{\min}(A) \geq 1 + (n-1)^2 2^{\frac{nH_{\max}^*(A)}{\sigma^*}} \geq 1 + (n-1)^2 > n-1, \tag{95}$$

due to $\frac{nH_{\max}^*(A)}{\sigma^*} \geq 0$ and $n \geq 2$. Thus, $\frac{A_{kk}}{1-A_{kk}} > n-1$. From (94) and (95), $f(x)$ is decreasing function and (93) is constructed by plugging the lower bound of A_{kk} in (89).

- Secondly, the lower bound of $\sigma_{\min}(A)$ can be found in [32] (Theorem 3)

$$\sigma_{\min}(A) \geq \min_{1\leq k\leq n} |A_{kk}| - \frac{1}{2}(R_k(A) + C_k(A)), \tag{96}$$

or in [33] (Theorem 0)

$$\sigma_{\min}(A) \geq \min_{1\leq k\leq n} \frac{1}{2}(\{4|A_{kk}|^2+(R_k(A)-C_k(A))^2\}^{1/2}-[R_k(A)+C_k(A)]), \tag{97}$$

with $R_k(A) = \sum_{j=1,j\neq k}^{j=n} |A_{kj}|$ and $C_k(A) = \sum_{j=1,j\neq k}^{j=n} |A_{jk}|$, respectively. Thus, if we use the lower bound established in (97),

$$\begin{aligned} \sigma_{\min}(A) &\geq \frac{1}{2}(\{4[\frac{c_{\min}(A)}{c_{\min}(A)+1}]^2\}^{1/2} \\ &- [\frac{1}{c_{\min}(A)+1} + \frac{n-1}{c_{\min}(A)+1}]) \\ &= \frac{c_{\min}(A)-n/2}{c_{\min}(A)+1} = \sigma^*, \end{aligned} \tag{98}$$

with (98) due to (89), (90), (91) and the fact that $\{R_k(A) - C_k(A)\}^2 \geq 0$.

A similar lower bound can be constructed using (96)

$$\begin{aligned} \sigma_{\min}(A) &\geq \frac{c_{\min}(A)}{c_{\min}(A)+1} \\ &- \frac{1}{2}(\frac{1}{c_{\min}(A)+1} + \frac{n-1}{c_{\min}(A)+1}) \\ &= \frac{c_{\min}(A)-n/2}{c_{\min}(A)+1} = \sigma^*, \end{aligned} \tag{99}$$

with (99) due to (89), (90) and (91). As seen, both our approaches yield a same lower bound of $\sigma_{\min}(A)$. However, (97) is tighter than (96) due to $\{R_k(A) - C_k(A)\}^2$. □

References

[1] Thomas M Cover and Joy A Thomas. *Elements of information theory*. John Wiley & Sons, 2012.
[2] Richard Blahut. Computation of channel capacity and rate-distortion functions. *IEEE transactions on Information Theory*, 18(4):460–473, 1972.
[3] Suguru Arimoto. An algorithm for computing the capacity of arbitrary discrete memoryless channels. *IEEE Transactions on Information Theory*, 18(1):14–20, 1972.
[4] Saburo Muroga. On the capacity of a discrete channel, mathematical expression of capacity of a channel which is disturbed by noise in its every one symbol and expressible in one state diagram. *Journal of the Physical Society of Japan*, 8(4):484–494, 1953.
[5] Claude Shannon. The zero error capacity of a noisy channel. *IRE Transactions on Information Theory*, 2(3):8–19, 1956.
[6] B Robert. Ash. information theory, 1990.
[7] Thuan Nguyen and Thinh Nguyen. On closed form capacities of discrete memoryless channels. In *2018 IEEE 87th Vehicular Technology Conference (VTC Spring)*, pages 1–5. IEEE, 2018.
[8] Keye Martin, Ira S Moskowitz, and Gerard Allwein. Algebraic information theory for binary channels. *Theoretical Computer Science*, 411(19):1918–1927, 2010.
[9] Xue-Bin Liang. An algebraic, analytic, and algorithmic investigation on the capacity and capacity-achieving input probability distributions of finite-input–finite-output discrete memoryless channels. *IEEE Transactions on Information Theory*, 54(3):1003–1023, 2008.
[10] Paul Cotae, Ira S Moskowitz, and Myong H Kang. Eigenvalue characterization of the capacity of discrete memoryless channels with invertible channel matrices. In *Information Sciences and Systems (CISS), 2010 44th Annual Conference on*, pages 1–6. IEEE, 2010.
[11] Mung Chiang and Stephen Boyd. Geometric programming duals of channel capacity and rate distortion. *IEEE Transactions on Information Theory*, 50(2):245–258, 2004.
[12] Michael Grant, Stephen Boyd, and Yinyu Ye. Cvx: Matlab software for disciplined convex programming, 2008.
[13] Abhishek Sinha. Convex optimization methods for computing channel capacity. 2014.
[14] Frédéric Dupuis, Wei Yu, and Frans MJ Willems. Blahut-arimoto algorithms for computing channel capacity and rate-distortion with side information. In *Information Theory, 2004. ISIT 2004. Proceedings. International Symposium on*, page 179. IEEE, 2004.
[15] Gerald Matz and Pierre Duhamel. Information geometric formulation and interpretation of accelerated blahut-arimoto-type algorithms. In *Information theory workshop, 2004. IEEE*, pages 66–70. IEEE, 2004.
[16] Yaming Yu. Squeezing the arimoto–blahut algorithm for faster convergence. *IEEE Transactions on Information Theory*, 56(7):3149–3157, 2010.
[17] Bernd Meister and Werner Oettli. On the capacity of a discrete, constant channel. *Information and Control*, 11(3):341–351, 1967.
[18] Masakazu Jimbo and Kiyonori Kunisawa. An iteration method for calculating the relative capacity. *Information and Control*, 43(2):216–223, 1979.
[19] Thai Duong, Duong Nguyen-Huu, and Thinh Nguyen. Location assisted coding (lac) embracing interference in free space optical communications. In *Proceedings of the 11th ACM Symposium on QoS and Security for Wireless and Mobile Networks*, pages 107–114, 2015.
[20] Claude E Shannon and Warren Weaver. *The mathematical theory of communication*. University of Illinois press, 1998.
[21] T Cover. An achievable rate region for the broadcast channel. *IEEE Transactions on Information Theory*, 21(4):399–404, 1975.
[22] Stephen Boyd and Lieven Vandenberghe. *Convex optimization*. Cambridge university press, 2004.
[23] Eric W Weisstein. Gershgorin circle theorem. 2003.
[24] Miroslav Fiedler and Vlastimil Pták. Diagonally dominant matrices. *Czechoslovak Mathematical Journal*, 17(3):420–433, 1967.
[25] Thuan Nguyen and Thinh Nguyen. Relay-miso channel. Available at http://ir.library.oregonstate.edu/concern/articles/tb09jb69h, 2018.
[26] Thomas Cover and A EL Gamal. Capacity theorems for the relay channel. *IEEE Transactions on information theory*, 25(5):572–584, 1979.
[27] Boris Rankov and Armin Wittneben. Achievable rate regions for the two-way relay channel. In *Information theory, 2006 IEEE international symposium on*, pages 1668–1672. IEEE, 2006.
[28] Stephen Boyd, Seung-Jean Kim, Lieven Vandenberghe, and Arash Hassibi. A tutorial on geometric programming. *Optimization and engineering*, 8(1):67, 2007.
[29] Fady Alajaji and Po-Ning Chen. *An Introduction to Single-User Information Theory*. Springer, 2018.
[30] Kaare Brandt Petersen, Michael Syskind Pedersen, et al. The matrix cookbook. *Technical University of Denmark*, 7(15):510, 2008.
[31] Rayleigh quotient and the min-max theorem. Available at http://www.math.toronto.edu/mnica/hermitian2014.pdf, 2014.
[32] Charles R Johnson. A gersgorin-type lower bound for the smallest singular value. *Linear Algebra and its Applications*, 112:1–7, 1989.
[33] YP Hong and C-T Pan. A lower bound for the smallest singular value. *Linear Algebra and its Applications*, 172:27–32, 1992.

Thuan Nguyen received a B.S. degree in Electrical Engineering (honors program) from Post and Telecommunication Institute of Technology, Vietnam, in 2013. He is currently a Ph.D. student at Oregon State University, Corvallis, Oregon, USA. His research interests include information theory, signal processing and machine learning.

Thinh Nguyen (M04) received the B.S. degree from the University of Washington, Seattle, WA, USA, in 1995 and the Ph.D. degree from the University of California, Berkeley, CA, USA, in 2003, both in electrical engineering. He is currently a Professor with the School of Electrical Engineering and Computer Science, Oregon State University, Corvallis, OR, USA. He is interested in all things stochastic, with applications to signal processing, distributed systems, wireless networks, network coding, and quantum walks.

Dr. Nguyen has served as an Associate Editor for the IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS FOR VIDEO TECHNOLOGY and the IEEE TRANSACTIONS ON MULTIMEDIA